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REVIEW

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ANALYTICAL METHODS OF SOLUTION OF BOUNDARY-VALUE PROBLEMS OF NONSTATIONARY HEAT CONDUCTION IN REGIONS WITH MOVING BOUNDARIES

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Classical linear problems of nonstationary heat conduction (and of related phenomena) for canonical regions and standard boundary conditions can be solved using well-developed analytical methods yielding an exact solution of the problem [1–19]. For a bounded region, its analytical solution in the form of a Fourier series where conjugation conditions for the functions in the boundary conditions of the problem at angular points of the phase region of determination of the equation of nonstationary heat conduction are not fulfilled [17] makes it possible to improve the convergence to an absolute and uniform one up to the boundary of the region [20–22]. The improved solutions become very convenient in consideration of many practical issues of thermophysics: calculations of thermophysical constants based on solution of inverse problems; determination of the time of heating of a canonically shaped workpiece; calculation of the time in which the process reaches the stationary phase, etc. In these and other cases, it becomes possible to investigate the kinetics of the processes based on calculational analytical relations of a parametric character.

The introduction of additional factors into the formulation of a boundary-value problem even of the linear type (complication of the shape of a body, motion of the boundary of the region, the time or space dependence of the thermophysical characteristics of a medium, etc.) necessitates the development of a special body of mathematics that, as a rule, yields an approximate solution of the problem and turns out to be efficient for obtaining an exact solution only in a certain situation.

Below, we will be dealing with the temperature fields in regions whose boundaries move with time. For parabolic equations of this kind, boundary-value problems are the subject of a practically unlimited number of investigations. As the years have gone by, their stream has not subsided, covering newer and newer substantive mathematical objects and an ever increasing number of various applications. Similar problems arise in the field of nuclear power engineering and safety of nuclear reactors [23–25]; in studying the process of combustion in solid-propellant rocket engines [26]; in using electric discharges and also in the phenomena of electric explosion of conductors [27] and other processes characterized by a high temperature (melting electric contacts [28], the action of an electric arc in contacts [29, 30], erosion of electric contacts [31, 32]); in a number of environmental and medicinal problems [33, 34]; in laser action on solids [35–43]; in phase transformations (the Stefan problem and the Verigin problem (in hydromechanics) with more complicated boundary conditions and a more general boundary-value problem for parabolic equations with a free boundary) [43–63], including the cases of freezing of the ground [64, 65], solidification of concrete [66], and freezing of solutions [67] and porous bodies [66]; in the processes of sublimation in freezing [69] and melting [70,

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71]; in the kinetic theory of crystal growth [72–81], in a number of thermomechanical problems (in heat shock [82–84], thermal decomposition [85, 86], and thermal shield of spacecraft [87]); in optimization theory [88] and numerical experiment [89]; in plasma dynamics and in plasma deposition [90–96]; in a number of issues of hydromechanics [97–100], filtration [101–105], and ablation [106–108] and strength of solids [109]; in mathematical modeling of physicochemical processes occurring with the movement of the interface in isotropic and anisotropic materials [110–117], including the processes indicated in [118, 119] (sublimation in vacuum, growth and collapse of a vapor bubble, evaporation of liquid droplets, electroslag remelting, use of the earth's heat, precipitation of erythrocytes, porous cooling, filtration of a solution in a polymer, action of superstrong magnetic fields, electroerosion grinding with a diamond-abrasive tool, oxidation of silicon, plastic metal working, combustion, formation of *p*-*n* transitions in lamellar semiconductors, some issues of the theory of dams, the mechanics of soils, the thermal behavior of oil pools, and also of filtration, electrodynamics, and elasticity [120]); in special-type thermal problems with an integral condition in studying the process of transfer of heat in a thin heated bar if the time-variable quantity of the heat of a part of the bar adjacent to one end of it is assigned [121, 122], etc.

The boundary-value problems of nonstationary transfer in a noncylindrical region (considered, in particular, in [23–122]) involve the cases where the motion of the boundary of the region is assigned or the cases where it is required that this motion be determined from additional conditions of the problem (the Stefan condition that expresses the energy balance in transition of the medium from one aggregate state to another or a Stefan-type condition in more general problems for a heat-conduction equation with a free boundary).

We consider below modern analytical methods of solution of boundary-value problems for parabolic equarions in noncylindrical regions, i.e., regions with curvilinear boundaries.

The abundance of works on boundary-value problems for parabolic equations in noncylindrical regions that refer to the methods of numerical solution (numerical experiment), analytical methods of finding exact and approximate solutions, asymptotic expansions of solutions, issues of qualitative theory, etc., required us to make of difficult decisions regarding the selection of material for the review and the list of publications on this subject. Moreover, the main task of the selection assumed that the primary focus would be on the methods of finding exact analytical solutions of boundary-value problems of nonstationary heat conduction in regions with boundaries moving with time, on functional constructions as analytical solutions for concrete laws of motion of a boundary, and on simple examples of an illustrative and substantive character, since they would help the reader to better understand the formulated analytical approaches and results. The purpose was to formulate a number of problems that required a solution.

In this connection, we consider in the work neither the issues of qualitative theory for parabolic equations nor asymptotic, numerical, graphical, and other approximate approaches. Nonetheless, in reflecting the role of outstanding domestic scientists in the development of different approaches to finding and investigating the solutions of boundary-value problems within the framework of differential equations of mathematical physics (and similar directions), these issues are also receiving proper attention.

The contemporary mathematical school of partial differential equations in Russia and the CIS countries has inherited a rich legacy. As early as in the 18th century, Russia was a country where mathematics made great progress. At that time, the St. Petersburg Academy of Sciences was joined by a new member – the eminent Euler, whose works in mathematics and mechanics are well known. In the first half of the 19th century, a significant contribution to the development of this science was made by Academicians M. B. Ostrogradskii, V. Ya. Bunyakovskii, and others. The great discovery of non-Euclidean geometry made by N. I. Lobachevskii enriched mathematics with new ideas that had a tremendous influence on the entire course of its subsequent development and that of related areas of science (including those connected with partial differential equations). Important work on the theory of differential equations and mechanics was done by S. V. Kovalevskaya. The 18th and 19th centuries should be called the period of the establishment and development of institutions of higher technical learning, including specialized schools of higher (for that time) type under the patronage of the Russian Academy of Sciences.

In the late 19th and early 20th centuries, the work of the scientists of the so-called St. Petersburg mathematical school headed by P. L. Chebyshev became widely known. It sparkled with such names as P. L. Chebyshev himself, A. N. Korkin, E. I. Zolotarev, A. A. Markov (Senior), A. M. Lyapunov, V. A. Steklov, G. F. Voronoi, D. A. Grave, and others. Their achievements in the field of mathematical physics and other divisions of mathematics are well known. In the early 20th century, a school of applied mathematics began to form in Moscow under the leadership of N. E. Zhukovskii and S. A. Chaplygin, and in the second decade of the 20th century a theoretical-functional school headed by D. F. Egorov and his disciple N. N. Luzin began to take shape. In 1920, the Institute of Physics and Mathematics was set up, with V. A. Steklov becoming its director. In 1923, the Institute of Physics and Mathematics was reorganized, with its mathematical division becoming a separate institution, soon to be transformed into the V. A. Steklov Institute of Mathematics (its first director being Academician I. M. Vinogradov), which became a major scientific mathematical center of the country. A number of Institutes of the Academy emerged on the basis of its departments, in particular, the Institute of Applied Mathematics (its first director being Academician M. V. Keldysh; now it is the M. V. Keldysh Institute of Applied Mathematics), and out of it in the 90s – the Institute of Mathematical Modeling (its first director being Academician A. A. Samarskii). Oming to the increase in scientific-research personnel, independent mathematical institutes or sectors of mathematics were set up at all Republic Academies of Sciences (i.e., National Academies of Sciences of the constituent Soviet Republics) and in many cities of the Russian Federation. Numerous scientific schools began to develop at the boundary between mathematics, mechanics, thermal physics, physics, chemistry, biology, etc.

A special role in the development of theoretical and applied thermal physics has been played by the Institute of Heat and Mass Transfer of the BSSR Academy of Sciences, now the Academic Scientific Complex "A. V. Luikov Institute of Heat and Mass Transfer," (IHMT), National Academy of Sciences of Belarus, and personally by Academician A. V. Luikov, Member of the BSSR Academy of Sciences. Aleksei Vasil’evich Luikov became head of the indicated Institute in 1956, and within a short period of time a small team of scientists grew to become a major thermophysical scientific center that became a Mecca of sorts for many decades for thermophysicists of all ages, levels, and ranks from different cities of the Soviet Union and foreign countries. A. V. Luikov initiated the All-Union Conferences on Heat and Mass Transfer that have been held every four years at the Institute since 1961. Since 1988 they have been International Forums; hundreds of scientists from different countries have participated in their work. The IVth International Forum on Heat and Mass Transfer held in May, 2000 was dedicated to the 90th anniversary of A. V. Luikov’s birth.

Another factor uniting thermophysicists of all levels has been the "Inzhenerno-Fizicheskii Zhurnal" (Inzh.-Fiz. Zh.), which was set up by A. V. Luikov in 1958 and led by him till the end of his life. It is hard to overestimate the role of the Inzh.-Fiz. Zh. and its influence on the development of thermal physics and, in particular, of the analytical theory of nonstationary heat and mass transfer through the publications of A. V. Luikov himself and his numerous disciples and followers. Thousands of young researchers have managed to achieve their personal fulfillment and to get a start in science owing to the Inzh.-Fiz. Zh., and, indeed, up to this time the A. V. Luikov IHMT and the "Inzhenerno-Fizicheskii Zhurnal," which have begun to be perceived as a united whole, have been playing a leading part in the preservation and development of the world’s thermophysical science.

O. G. Martynenko, Member of the National Academy of Sciences and editor-in-chief of the Inzh.-Fiz. Zh., dedicated a detailed paper, A. V. Luikov’s Scientific Legacy (On the 90th Anniversary of His Birth) [*Inzh.-Fiz. Zh.*, **73**, No. 5, 869–883 (2000)], to A. V. Luikov’s life and work and to his enormous scientific legacy.

Owing to their extremely wide application, the classical boundary-value problems for differential equations of mathematical physics historically have drawn the attention of scientists of different directions:

mathematicians, mechanics, physicists, chemists, thermophysicists, etc. New, more general and more correct, physical models and corresponding mathematical models of processes have been created; new analytical, graphical, numerical (using the finite-difference method) methods, methods of analogies, and other approaches for the solution of entire classes of problems have been developed; the development of the qualitative theory of partial differential equations has reached an extremely high level. The use of the computer-based numerical methods has significantly expanded the class of mathematical models that allow exhaustive analysis. Based on an exact solution of a problem even of a cumbersome form, one has been able to track the influence of any parameter on the kinetics of a process. Difference schemes of approximate calculation of a problem's solution [123, 124] make it possible not to seek substantial simplifications that are necessary to obtain an exact analytical solution in constructing an initial mathematical model of the process. The qualitative theory of partial differential equations makes it possible, without solving the differential equations themselves (with assigned boundary conditions), to obtain the required information on the solution's respective properties [125–127] (including those for boundary-value problems in noncylindrical regions [128–135]).

The analytical methods of the theory of nonstationary transport make it possible to obtain solutions of a great number of boundary-value problems. The results of such solutions allow a clear and convenient analysis of phenomena and make it possible to reflect the influence of all factors, to assess their significance, and to identify the main ones among them. The presence of analytical solutions of a certain class of boundary-value problems is also of interest for the construction of difference schemes of approximate calculation of solutions for rather complex problems that are not readily investigated by other methods. The certainty of a solution being calculated correctly is achieved by the use of the same computational scheme for calculating model problems whose exact analytical solutions are known in advance.

In the past years, outstanding scientists of the former Soviet Union and Russia made their contribution to the development of mathematics and its applications in terms of corresponding model representations. It is impossible to give a sufficiently full list of the entire galaxy of scientists who have worked and are currently working in these areas within the scope of a review paper. Recognizing all this, the author has confined himself to the publications cited in the References.

From the mathematical point of view, the boundary-value problems for parabolic equations in the region with a moving boundary are fundamentally different from classical ones (for cylindrical regions). Owing to the dependence of the region's boundary on time, the methods of separation of variables and of integral Fourier–Hankel–Laplace transforms [14–21] are not applicable to this class of problems in the general case, since, remaining within the framework of the classical methods of mathematical physics [136–153], one is unable to coordinate the solution of the heat-conduction equation with the motion of the boundary of the heat-transfer region. The development of this problem seems to have proceeded in the following way. On the one hand, it became possible to obtain exact solutions of problems of this type using apt guesswork and artificial procedures, and for a quite limited number of cases of boundary motion (first a linear one in the region $x \geq l + vt,\ t \geq 0$, then parabolic $x \geq \beta\sqrt{t},\ t \geq 0$, [119]) and for a particular form of boundary conditions: constant ones and those of the first kind. On the other hand, these problems were used, granted their quite general formulation, to perfect the classical methods of solution of boundary-value problems for differential equations of mathematical physics (and their modification) [154]: thermal potentials; contour integration; extensions; the Green's functions; variational methods; series expansion of the function sought in generalized powers; generalization of finite integral transforms to noncylindrical regions; Grinberg's functional transforms, and methods based on the use of integral, integro-differential, or ordinary differential equations, difference, asympotic, and numerical ones [155–186]. It was also explicable why different approaches were used to solve one and the same class of problems. This can be explained by the fact that the solution of one and the same thermal problem can be sought in different classes of functions that are determined by the analytical approach in solving the problem. These functions must be such that they, first, could be found quite easily and, second, would ensure the convergence on the process so well that it could be possible to draw conclusions on the properties of the obtained solution that are required by the problem. The representation of the analytical solution of the problem in equivalent functional forms (identical in the sense of number) is of great practical value since it allows variation of the solution depending on the problem's formulation: for example, a solution in the form of a Fourier-type series that is convenient for large times (found by an integral Fourier transformation), or in the form of Poisson's summation formula more suitable for small times (found by an operational method). For regions with moving boundaries this fact is of particular importance, taking into account the above-mentioned widespread use of problems of this class. At the same time, it should be emphasized that, despite the well-developed analytical theory of nonstationary heat and mass transfer and close directions [1–21, 187–213], the success achieved in the last two decades in finding exact analytical solutions of the problems for different laws of motion of a boundary is very insignificant. Technical difficulties of a computational character in finding an exact analytical solution of the problem and loss of attention of the analytical transport theory to this region are, apparently, one reason for this situation. At the same time, the qualitative theory of parabolic differential equations in noncylindrical regions has forged far ahead during this time.

The formulation of the boundary-value problem of nonstationary heat conduction considered in the review is as follows.

Let Ω_t be a noncylindrical region in the phase space $(n+1)$ of measurements whose section by the plane-characteristic t = const $\geq t_0 > 0$ is a convex region D_t ($D_t \in R^n$) of variation of $M(x_1, x_2, ..., x_n)$, S_t be a piecewise-smooth surface dependent on the time $t \geq 0$ and limiting the region D_t, and $\mathbf{n}$ be the external normal to S_t so that $\overline{\Omega}_t = \{M \in \overline{D}_t = D_t + S_t,\ t \geq 0\}$. Let $T(M, t)$ be a temperature function satisfying the conditions of the problem:

$$\frac{\partial T}{\partial t} = a\Delta T(M, t) + f(M, t)\,, \quad M \in D_t\,, \quad t > 0\,; \tag{1}$$

$$T(M, t)\big|_{t=0} = \Phi_0(M)\,, \quad M \in \overline{D}_{t=0}\,; \tag{2}$$

$$\beta_1 \frac{\partial T(M, t)}{\partial n} + \beta_2 T(M, t) = \varphi(M, t)\,, \quad M \in S_t\,, \quad t \geq 0\,. \tag{3}$$

Here

$$f(M, t) \in C^0(\overline{\Omega}_t)\,; \quad \Phi_0(M) \in C^1(\overline{\Omega}_t)\,; \quad \varphi(M, t) \in C^0(S_t \times t \geq 0)\,; \quad \beta_1^2 + \beta_2^2 > 0\,.$$

The solution sought is

$$T(M, t) \in C^2(\Omega_t) \cap C^0(\overline{\Omega}_t)\,, \quad \text{grad}_M T(M, t) \in C^0(\overline{\Omega}_t)\,. \tag{4}$$

Although the presented approaches hold true for boundary-value problems for Eq. (1), in fact, we can also consider equations of the form (for $n = 3$)

$$\frac{\partial T}{\partial t} = a\Delta T(M, t) - b^2 T(M, t) + \mathbf{v} \cdot \text{grad}\, T(M, t) + f(M, t)\,, \tag{5}$$

since by the formulation

$$T(M,t) = U(M,t)\exp\left[-\frac{1}{2a}\mathbf{r}\cdot\mathbf{v} - \left(b^2 + \frac{1}{4a}\sum_{i=1}^{3} v_i^2\right)t\right],$$

where $M = M(x_1, x_2, x_3)$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ (v_i = const), b^2 = const, and $\mathbf{r} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, Eq. (5) is reduced to the case (1)

$$\frac{\partial U}{\partial t} = a\Delta U(M,t) + F(M,t)\ .$$

Here $F(M, t)$ is a new (known) function.

Considered below are predominantly linear thermal problems (n = 1) in a Cartesian coordinate system and plane problems (n = 2) in cylindrical (a cylindrical field) and spherical (a spherical field) coordinate systems – the most extensively studied cases at present. As far as spatial regions are concerned, the solution of a multidimensional problem for canonical regions can be represented in the form of a product of solutions of one-dimensional problems (for example, in constructing the Green's function). The method of functional transformations considered below also makes it possible to study (1)–(3) in spatial regions, retaining and not retaining similarity.

In regions with moving boundaries, just as in the case of cylindrical regions, we can also speak of the first (β_1 = 0), the second (β_2 = 0), or the third ($\beta_i > 0$, i = 1, 2) boundary-value problems. However, the indicated equivalence in representation of boundary conditions is not always retained. In particular, the heat-insulation condition for a moving boundary of the region $x \in [0, y(t)]$, $t \geq 0$, where $y(t)$ for $t > 0$ is a continuously differentiable function with finite derivatives of any order, has the form [154]

$$\left[\frac{\partial T(x,t)}{\partial x} + \frac{v(t)}{a}T(x,t)\right]_{x=y(t)} = 0\ ,\quad t > 0\ , \tag{6}$$

and for the velocity of motion $v(t) = dy(t)/dt = 0$ ($y(t)$ = const) expression (6) coincides with a classical representation of the heat insulation of an immovable (fixed) boundary that follows from the Fourier law in scalar form [16]. The special properties of the region with a moving boundary also manifest themselves in the formulation of boundary-value problems for the corresponding Green's functions [214]. Here, special attention should be concentrated on finding the Green's function for the second and third boundary-value problems (see below).

With regard to each of the boundary-value problems (1)–(3) there arise issues of correctness of their formulation: (1) the existence of the solution; (2) the uniqueness of the solution, and (3) the stability of the solution. As has been indicated, these issues are considered in the qualitative theory of parabolic equations in regions with curvilinear boundaries. Problems not satisfying the enumerated requirements (1)–(3) above are called incorrectly formulated. In 1962, A. N. Tikhonov developed new approaches to solving incorrectly formulated problems whose basis was formed by the fundamental notion of a regularizing operator [215]. These issues have also been the object of the investigations [216–218], and also of the works on inverse problems of heat conduction that are incorrect in formulation [219–221].

It is assumed below that the boundary functions in (1)–(3) and the laws of movement of a boundary are smooth functions for which all the transformations occurring in the process of calculation exist. These functions are prescribed by the practice of numerous applications of problems (1)–(3). Analytical solutions of the latter belong to the class of functions (4). In the 1930s, S. L. Sobolev developed a theory of generalized solutions for partial differential equations. Subsequently the qualitative theory of boundary-value problems for parabolic equations in a generalized formulation in Sobolev spaces and other functional spaces and the theory of these spaces for the solution of problems (the embedding theorems, the trace theorems, compactness of embedding and the theory of averaging, etc.) [125–127, 222–225] and also other theories were developed. These issues are considered in detail in the recently published work [146].

1. Method of Green's Functions. Regions $x \in [y_1(t), y_2(t)]$, $t \geq 0$, and $x \in [y(t), \infty)$, $t \geq 0$

The method of Green's functions is more preferable than other approaches in view of its universality. It can be applied to solution of problems (1)–(3) in one-, two-, and three-dimensional cases in bounded and semibounded regions Ω_t for rather general boundary functions in (1)–(3) and source functions. Each case of finding the Green's function of the boundary-value problems (1)–(3) is very important, since it contains voluminous information, permitting one to write the integral form of a great number of analytical solutions depending on the inhomogeneities in (1)–(3).

In [226], it is established that for the problem

$$\partial T/\partial t = a\partial^2 T/\partial x^2 + f(x, t), \quad y_1(t) < x < y_2(t), \quad t > 0; \tag{7}$$

$$T(x, 0) = \Phi_0(x), \quad y_1(0) \leq x \leq y_2(0); \tag{8}$$

$$[\beta_{i1}\partial T(x, t)/\partial x + \beta_{i2} T(x, t)]_{x=y_i(t)} = \beta_{i3}\varphi_i(t), \quad t \geq 0, \tag{9}$$

the corresponding Green's function $G(x, t, x', \tau)$ as a function of (x, t) is found from the conditions

$$\partial G/\partial t = a\partial^2 G/\partial x^2, \quad y_1(t) < x < y_2(t), \quad t > \tau; \tag{10}$$

$$G(x, t, x', \tau)|_{t=\tau} = \delta(x - x'), \quad y_1(\tau) < (x, x') < y_2(\tau); \tag{11}$$

$$(\beta_{i1}\partial G/\partial x + \beta_{i2} G)_{x=y_i(t)} = 0, \quad t > \tau \quad (i = 1, 2), \tag{12}$$

and as a function of (x', τ) satisfies the conditions

$$\partial G/\partial \tau + a\partial^2 G/\partial x'^2 = 0, \quad y_1(\tau) < x' < y_2(\tau), \quad \tau < t; \tag{13}$$

$$G(x, t, x', \tau)|_{\tau=t} = \delta(x' - x), \quad y_1(t) < (x', x) < y_2(t), \tag{14}$$

and then in the case of the first boundary-value problem in (9) ($\beta_{i1} = 0$; $\beta_{i2} = \beta_{i3} = 1$)

$$G(x, t, x', \tau)|_{x'=y_i(\tau)} = 0, \quad \tau < t \quad (i = 1, 2); \tag{15}$$

in the case of the second boundary-value problem in (9) ($\beta_{i2} = 0$; $\beta_{i1} = \beta_{i3} = 1$)

$$\left[\frac{\partial G}{\partial x'} + \frac{1}{a}\frac{dy_i(\tau)}{d\tau} G\right]_{x'=y_i(\tau)} = 0, \quad \tau < t \quad (i = 1, 2); \tag{16}$$

in the case of the third boundary-value problem in (9) ($\beta_{i1} = 1$; $\beta_{i2} = \beta_{i3} = (-1)^i h_i$)

$$\left\{\frac{\partial G}{\partial x'} + (-1)^i \left[h_i + (-1)^{i-1} \frac{1}{a} \frac{dy_i(\tau)}{d\tau}\right] G\right\}_{x'=y_i(\tau)} = 0\,, \quad \tau < t \quad (i = 1, 2)\,. \tag{17}$$

Here $\delta(z)$ is the Dirac delta function.

Thus, the function $G(x, t, x', \tau)$ can be found as the solution of equivalent problems for Eqs. (10) and (13) with the above boundary conditions. The main conclusion is that for regions with moving boundaries no equivalence is preserved in representation of boundary conditions in the formulations of the problems in (x, t) and (x', τ) unlike cylindrical regions. Thus, in the case of the second and third boundary-value problems in (7)–(9), construction of the Green's functions from (x', τ) is related to the time-variable "relative coefficient of heat exchange" in the boundary conditions (16)–(17). This is a rather cumbersome class of problems, to which item 7 of the review is devoted. In practice, it is expedient to use the formulation (11)–(12) in constructing the Green's functions. For a region with moving boundaries, the function $G(x, t, x', \tau)$, owing to its physical meaning (the thermal pulse of the power $Q = c\rho$ [139]), depends on t and τ and not on the difference $(t - \tau)$, since it is not just the action time $(t - \tau)$ but also the instant τ of occurrence of the pulse that will be determining. The integral representation of the analytical solution of problem (7)–(9) has the form

$$T(x, t) = \int_{y_1(0)}^{y_2(0)} T(x', 0)\, G(x, t, x', 0)\, dx' + \int_0^t \int_{y_1(\tau)}^{y_2(\tau)} f(x', \tau)\, G(x, t, x', \tau)\, d\tau dx' +$$

$$+ \sum_{i=0}^{2} \int_0^t \left\{\left[\alpha_{i1} \frac{\partial T(x', \tau)}{\partial x'} + \alpha_{i2} T(x', \tau)\right] \times \left[\gamma_{i1} \frac{\partial G(x, t, x', \tau)}{\partial x'} + \gamma_{i2} G(x, t, x', \tau)\right]\right\}_{x'=y_i(\tau)} d\tau\,, \tag{18}$$

where

$\alpha_{i1} = \gamma_{i2} = 0$; $\alpha_{i2} = (-1)^{i-1}$, $\gamma_{i1} = 1$ in the case of the first boundary-value problem;

$\alpha_{i2} = \gamma_{i1} = 0$; $\alpha_{i1} = (-1)^i$; $\gamma_{i2} = 1$ in the case of the second boundary-value problem;

$\alpha_{i1} = (-1)^i$; $\alpha_{i2} = h_i$; $\gamma_{i1} = 0$, and $\gamma_{i2} = 1$ in the case of the third boundary-value problem.

From (18) it is easy to obtain a similar representation for the region $[y(t), \infty)$, $t \ge 0$, too, letting $x' = y_2(\tau) \to \infty$ and taking into account that the functions T and G and their derivatives, with respect to x' tend to zero. In finding a particular expression for the function G, it is recommended in [214] that $G(x, t, x', \tau)$ be represented in the form

$$G(x, t, x', \tau) = \frac{1}{2\sqrt{\pi a (t - \tau)}} \exp\left[-\frac{(x - x')^2}{4a(t - \tau)}\right] + q(x, t, x', \tau) = G_0 + q\,, \tag{19}$$

where G_0 is the fundamental solution of Eq. (7) (for $f = 0$); $q(x, t, x^I, \tau)$ is the regular component of the Green's function to be found from the problem (10)–(12) transformed in advance relative to the function q with a homogeneous initial condition.

The above method held true for boundary-value problems for the parabolic equation (1), and in this respect it is quite a developed theory. The situation is much worse for a hyperbolic heat-conduction equation of the type of [1] in a noncylindrical region:

$$\frac{1}{a} \frac{\partial T(x, t)}{\partial t} + \frac{1}{v_h^2} \frac{\partial^2 T}{\partial t^2} = \frac{\partial^2 T}{\partial x^2} + F(x, t)\,, \quad y_1(t) < x < y_2(t)\,, \quad t > 0\,, \tag{20}$$

where $v_h = \sqrt{a/\tau_{rel}}$, is the velocity of propagation of heat and τ_{rel} is the relaxation time of the heat flux. For this case, the method of the Green's functions is practically not developed, and the problem remains open. A similar situation also occurs for boundary-value problems based on the Gurtin–Pipkin integro-differential equation of heat conduction with allowance for thermal memory [191]. Thus, in the case of thermal heating (the second boundary-value problem) the indicated model has the following form:

$$\beta(0)\frac{\partial T(x,t)}{\partial t} + c_v\frac{\partial^2 T}{\partial t^2} + \int_0^\infty \beta'(\tau)\frac{\partial T(x,t-\tau)}{\partial t}d\tau =$$

$$= \alpha(0)\frac{\partial^2 T}{\partial x^2} + \int_0^\infty \alpha'(\tau)\frac{\partial^2 T(x,t-\tau)}{\partial x^2}d\tau\,, \quad 0 < x < y(t)\,, \quad t > 0\,;$$

$$T(x,0) = [\partial T(x,t)/\partial t]_{t=0} = 0\,, \quad 0 \le x \le y(0)\,;$$

$$\int_0^\infty \alpha(\tau)\frac{\partial T(x,t-\tau)}{\partial x}\bigg|_{x=0} d\tau = q_0\,, \quad t > 0\,; \tag{21}$$

$$\int_0^\infty \alpha(\tau)\frac{\partial T(x,t-\tau)}{\partial x}\bigg|_{x=y(t)} = q_1\,, \quad t > 0\,,$$

where c_v is the volumetric heat capacity and $\alpha(t)$ and $\beta(t)$ are, respectively, the (selected) relaxation functions of the heat flux and the internal energy. The development of analytical theory (exact solutions; qualitative issues) for this class of boundary-value problems is one promising method of the modern analytical theory of nonstationary heat transfer.

2. Method of Thermal Potentials. Regions $x \in [l + vt, \infty)$, $t \ge 0$, and $x \in [0, l + vt]$, $t \ge 0$. Green's Functions

In [139, 148], generalized thermal potentials of a single and double layer are described as one possible body of analytics in solving problems (7)–(9). It is shown that their application reduces the problem to either a Volterra integral equation of the second kind or a system of such equations that always possesses a solution. Using the Picard process, we were able to write the first (one or two) approximations of the solution for this system. The intricacy and cumbersomeness of the indicated procedure in using thermal potentials has repeatedly been noted in the literature. In [16, 84], this method was modified and turned out to be especially efficient for regions with a uniformly moving boundary, both in solving the problem in the initial formulation (7)–(9) (for $f = 0$ or $f = f(t)$) and in constructing the corresponding Green's function. The resultant analytical solutions of the problem have a new (simpler) integral form different from those known earlier. The solution of the problem

$$\partial T/\partial t = a\partial^2 T/\partial x^2\,, \quad x > l + vt\,, \quad t > 0\,; \tag{22}$$

$$T(x,0) = 0\,, \quad x \ge l\,; \quad (\beta_1 \partial T/\partial x + \beta_2 T)_{x=l+vt} = \beta_3 \varphi(t)\,, \quad t \ge 0\,; \tag{23}$$

$$|T(x,t)| < +\infty\,, \quad x \ge l + vt\,, \quad t \ge 0\,, \tag{24}$$

is written in the form of the generalized thermal potential of a single layer relative to the curve $x = l + vt$:

$$T(x, t) = \frac{\sqrt{a}}{2\sqrt{\pi}} \int_0^t \frac{\Psi(\tau)}{\sqrt{t-\tau}} \exp\left[-\frac{(x-l-v\tau)^2}{4a(t-\tau)}\right] d\tau, \tag{25}$$

where $\Psi(t)$ is the unknown potential density to be found from the boundary condition (23). In the space of Laplace transforms $\overline{T}(x, p) = \int_0^\infty T(x, t) \exp(-pt) dt$, Re $p \ge \beta > 0$, $|\arg p| < \frac{\pi}{2}$, expression (25) acquires the form

$$\overline{T}(x, p) = \frac{\sqrt{a}}{2\sqrt{p}} \exp\left[-(x-l)\sqrt{p/a}\right] \overline{\Psi}\left(p - v\sqrt{p/a}\right), \tag{26}$$

whence it follows that the unknown density should be sought relative to the form $\overline{\Psi}(p - v\sqrt{p/a})$. The final operational (basic) solution of problem (22)–(24) has the form

$$\overline{T}(x, p) = \overline{\Theta}(p)\left[1 - \frac{v/(2a)}{\sqrt{p/a}}\right] \exp\left[-(x-l)\sqrt{p/a}\right] \overline{\varphi}\left(p - v\sqrt{p/a}\right), \tag{27}$$

where $\overline{\Theta}(p) = \begin{cases} 1 & \text{for the first boundary-value problem } (\beta_1 = 0,\ \beta_2 = \beta_3 = 1); \\ -1/\sqrt{p/a} & \text{for the second boundary-value problem } (\beta_2 = 0,\ \beta_1 = \beta_3 = 1); \\ h/(h + \sqrt{p/a}) & \text{for the third boundary-value problem } (\beta_1 = 0,\ \beta_2 = \beta_3 = -h). \end{cases}$

Expression (27) involves numerous particular cases of the boundary function $\varphi(t)$ in (23) that are of practical interest: homogeneous, pulsed, pulsating, periodic, etc. Passage to the inverse transforms occurs by the known rules of operational calculus and leads to analytical solutions of a very compact form. Thus, in the case of the first boundary-value problem

$$T(x, t) = \frac{1}{2\sqrt{\pi a}} \int_0^t \frac{x - (l + vt)}{(t-\tau)^{3/2}} \varphi(\tau) \exp\left[-\frac{(x-l-v\tau)^2}{4a(t-\tau)}\right] d\tau. \tag{28}$$

All the reasoning also holds in the presence of the homogeneous nonstationary source $f(t)$ in (22), and on the right-hand side in (28) there appears a term $\int_0^t f(\tau) d\tau$ that makes the calculations slightly more complicated. In the case of a nonhomogeneous nonstationary source $f(x, t)$ and a nonhomogeneous boundary condition, we should construct the Green's function in advance, using for this purpose expression (27), the above definition of the Green's function, and its representation in the form (19). In the case of the third boundary-value problem [226],

$$G(x, t, x', \tau) = \frac{1}{2\sqrt{\pi a(t-\tau)}} \left\{ \exp\left[-\frac{(x-x')^2}{4a(t-\tau)}\right] + \exp\left[-\frac{(x+x'-2(l+v\tau))^2}{4a(t-\tau)} + \frac{v}{a}(x' - (l+v\tau))\right]\right\} -$$

$$-\left(h + \frac{v}{2a}\right) \exp\left\{[x + x' - 2(l + v\tau)]h + ah^2(t-\tau) + \frac{v}{a}[x' - (l+v\tau)]\right\} \times$$

$$\times \Phi^* \left[\frac{x + x' - 2\,(l + v\tau)}{2\,\sqrt{a\,(t - \tau)}} + h\,\sqrt{a\,(t - \tau)} \right], \tag{29}$$

where $\Phi^*(z) = 1 - \Phi(z)$ and $\Phi(z) = (2/\sqrt{\pi}) \int_0^z \exp(-y^2) dy$ is the Laplace function. Setting in (29) $h = 0$, we find the Green's function of the second boundary-value problem; passage to the limit when $(1/h) \to 0$ leads to Green's function for the first boundary-value problem. The given relations make it possible to consider the case of spherical symmetry, too, if it is remembered that the equation

$$\frac{\partial T}{\partial t} = a \left(\frac{\partial^2 T}{\partial r^2} + \frac{2}{r} \frac{\partial T}{\partial r} \right), \quad r > R + v\tau\,, \quad t > 0\,, \tag{30}$$

using the substitution $U(r, t) = rT(r, t)$ is reduced to the form (22). In the case of the cylindrical field $T(r, t)$ in $r > R + v$, $t > 0$, it is expedient to use the method of functional transformations considered below. The theory of thermal potentials is practically not developed for this case.

As to the bounded region $x \in [0, l + vt]$, $t \geq 0$, for Eq. (22) with a homogeneous initial condition, here it suffices to use the thermal potential of a single layer relative to the curve $x = 0$ and the generalized thermal potential of a single layer relative to the curve $x = l + vt$ with boundary conditions of any kind [226]:

$$T(x, t) = \frac{\sqrt{a}}{2\sqrt{\pi}} \int_0^t \frac{\Psi_1(\tau)}{\sqrt{t - \tau}} \exp\left[-\frac{x^2}{4a\,(t - \tau)} \right] d\tau + \frac{\sqrt{a}}{2\sqrt{\pi}} \int_0^t \frac{\Psi_2(\tau)}{\sqrt{t - \tau}} \exp\left[-\frac{(x - l - v\tau)^2}{4a\,(t - \tau)} \right] d\tau\,, \tag{31}$$

where $\Psi_i(t)$ $(i = 1, 2)$ are the unknown densities to be found from the boundary conditions relative to the form that is established in the space of Laplace transforms

$$\overline{T}(x, p) = \frac{1}{2\sqrt{p/a}} \left\{ \overline{\Psi}_1(p) \exp(-x\sqrt{p/a}) + \overline{\Psi}_3 \left[\left(\sqrt{p} + \frac{v}{2\sqrt{a}} \right)^2 \right] \exp[-(l - x)\sqrt{p/a}\,] \right\}, \tag{32}$$

where $\Psi_3(t) = \Psi_2(t) \exp(v^2 t/4a)$. The method leads to new integral representations for the analytical solutions of thermal problems in the region with a uniformly moving boundary. For example, with boundary conditions of the first kind in (9) the method yields

$$T(x, t) = \frac{1}{2\sqrt{a\pi}} \sum_{n=-\infty}^{n=+\infty} \sum_{k=0}^{1} (-1)^k \int_0^t \frac{[x + (2n + k)]\,(l + vt)}{(t - \tau)^{3/2}} \varphi_{k+1}(\tau) \times$$

$$\times \exp\left\{ -\frac{v\,(l + v\tau)\,(n^2 + kn)}{a} - \frac{[x + (2n + k)\,(l + v\tau)]^2}{4a\,(t - \tau)} \right\} d\tau\,. \tag{33}$$

In the presence of the inhomogeneities indicated in (7)–(9), the problem becomes rather difficult technically; however relation (33) makes it possible to easily overcome difficulties in finding the corresponding Green's function. In the space of transforms, expression (33) has the form

$$\overline{T}(x, p) = \overline{\varphi}_1(p) \exp(x\sqrt{p/a}\,) + \frac{1}{\sqrt{p}} \sum_{n=0}^{\infty} \sum_{k=0}^{1} (-1)^k [\sqrt{p} + \gamma\,(2n + k)] \times$$

$$\times \exp\left[-\frac{2l\gamma}{\sqrt{a}}(n^2+nk)\right]\left\{\exp\left[-\frac{(2n+k)\,l+x}{\sqrt{a}}\sqrt{p}\right]-\exp\left[-\frac{(2n+k)\,l-x}{\sqrt{a}}\sqrt{p}\right]\right\}\bar{\varphi}_{2k+1}\,[(\sqrt{p}+(2n+k)\,\gamma)^2]\,, \tag{34}$$

where $\varphi_3(t) = \varphi_2(t)\exp(\gamma^2 t)$ and $\gamma = v/(2\sqrt{a})$. Expression (34) can serve as the working formula for writing the analytical solutions (in transforms) of the first boundary-value problem for a wide class of the boundary functions $\varphi_i(t)$ (for $\Phi_0 = f = 0$), including the construction of the Green's function, based on representation (19) (in passage to the inverse transforms in (34), we should single out in advance the term $n = 0$ for $k = 0$). For $G(x, t, x', \tau)$ we have (in the case of the first boundary-value problem)

$$G(x, t, x', \tau) = \frac{1}{2\sqrt{\pi a\,(t-\tau)}}\sum_{n=-\infty}^{n=+\infty}\exp\left(-\frac{2l_0\,\gamma n^2+2\gamma x' n}{\sqrt{a}}\right)\sum_{k=1}^{2}(-1)^{k-1}\exp\left[-\frac{(2l_0 n+x'+(-1)^k x)^2}{4a\,(t-\tau)}\right], \tag{35}$$

where $l_0 = l + v\tau$ and $\gamma = v/2\sqrt{a}$. Knowing the Green's function (35), we can write the analytical solution of the first boundary-value problem in (7)–(9) in integral form using (18). For example, in the case $\Phi_0(x) = T_0$, $\varphi_i(t) = T_w$ $(i = 1, 2)$ expression (18) yields $(f = 0)$

$$W(z, \mathrm{Fo}) = \frac{T(x, t) - T_w}{T_0 - T_w} = \frac{1}{2\sqrt{\pi \mathrm{Fo}}}\sum_{n=-\infty}^{n=+\infty}\exp(-v_0 n^2)\times$$

$$\times\int_0^1 \exp(-v_0 n\xi)\left[\exp\left(-\frac{(2n+\xi-z)^2}{4\mathrm{Fo}}\right)-\exp\left(-\frac{(2n+\xi+z)^2}{4\mathrm{Fo}}\right)\right]d\xi\,, \tag{36}$$

where $z = x/l$, $v_0 = vl/a$, and $\mathrm{Fo} = at/l^2$.

As to the region $x \in [l_1 + v_1 t,\ l_2 + v_2 t]$, $t \ge 0$, for $T(x, t)$ in (7)–(9), this case is easily reduced to the previous one using the following transformations:

$$z = x - (l_1 + v_1 t)\,, \quad T(x, t) \equiv W(z, t)\,; \quad W(z, t) = \Theta(z, t)\exp(-v_1 z/2a - v_1^2 t/4a)\,, \tag{37}$$

where $z \in [0,\ l_0 + v_0 t]$, $t \ge 0$; $l_0 = l_2 - l_1$ and $v_0 = v_2 - v_1$; $\Theta(z, t)$ satisfies Eq. (7) (for a new (known) source function). Similarly, we can also consider the region $x \in [vt,\ 1 + vt]$, $t \ge 0$; the transformations (37) reduce this region to the case $x \in [0, l]$, $t > 0$, studied in detail in [20]. For the case of central symmetry (30) for $r \in [0,\ R + vt]$, $t \ge 0$, the Green's function of the first boundary-value problem has the form

$$G(r, t, r', \tau) = \frac{1}{8\pi r r'\sqrt{\pi a\,(t-\tau)}}\sum_{n=-\infty}^{n=+\infty}\exp\left(-\frac{R_0 v n^2 + r' v n}{a}\right)\sum_{k=1}^{2}(-1)^{k+1}\exp\left[-\frac{(2R_0 n + r' + (-1)^k r)^2}{4a\,(t-\tau)}\right],$$

where $R_0 = R + v\tau$. For other boundary conditions, the specific features of the method (31)–(32) lie in just solving a finite-difference equation of the $\bar{F}(p + b) - \bar{F}(p) = \bar{C}(p)$ type in finding the unknown densities of potentials in (32) and in passage to the inverse transform. In the type of boundary conditions, problem (7)–(9) allows nine formulations in the region $x \in [0,\ l + vt]$, $t \ge 0$, and the same number of formulations for the case of central symmetry. Studied in literature are cases 1–1; 1–2; 1–3; 2–1 and, in a very cumbersome form, case 3–3 (references in [119]). Although a uniform law of motion of a boundary is the most frequently used in applications [35, 154], nonetheless, the results accumulated show that the problem invites further study: it is required that the Green's functions and temperature-distribution curves be constructed from solutions of the type (36) and effects that can appear in motion of the region boundary be studied [16]. An important element of investigations in this problem is finding the Green's function $G(x, t, x', \tau)$ in different functional forms for large and small times. For example, for the region $x \in [vt, l+vt]$, $t \geq 0$, in the case of the first boundary-value problem we can write

$$G(x, t, x', \tau) = \frac{1}{2\sqrt{\pi a (t-\tau)}} \exp\left\{-\frac{v}{2a}\left[(x-x') - (v/2)(t-\tau)\right]\right\} \times$$

$$\times \sum_{n=-\infty}^{n=+\infty} \left\{\exp\left[-\frac{(2\ln + x - x' - v(t-\tau))^2}{4a(t-\tau)}\right] -\right.$$

$$\left. - \exp\left[-\frac{(2\ln + x + x' - v(t+\tau))^2}{4a(t-\tau)}\right]\right\} = \frac{2}{l}\exp\left[\frac{v}{2a}(x'-x) + \frac{v^2}{4a}(t-\tau)\right] \times$$

$$\times \sum_{n=1}^{\infty} \sin\frac{n\pi(x'-v\tau)}{l} \sin\frac{n\pi(x-vt)}{l} \exp\left[-\left(\frac{n\pi\sqrt{a}}{l}\right)^2 (t-\tau)\right].$$

This problem is open even for the classical region $x \in [0, l]$, $t \geq 0$, where only the simplest cases (1–1), (1–2), and (2–1) of the nine are considered. Practically not considered are the regions $r \in [R_1, R_2]$, $t \geq 0$ ($R_1 \geq 0$, $R_2 > 0$) for spherical and cylindrical symmetry, to say nothing of more complicated regions (with a uniformly moving boundary) [16]. As is seen, even the simplest linear problems of the analytical theory of heat conduction are far from being exhausted as the subject of investigations. As to the region $r \in [0, R+vt]$, $t \geq 0$, for the case of the radial heat flux in cylindrical coordinates, the theory of thermal potentials here remains to be developed. For this case we can use a generalized Appel transformation (references in [119]). The inversion-type transformation

$$z = \frac{\gamma(r+\alpha)}{t+\beta} + c_1\,; \quad \tau = -\frac{\gamma^2}{t+\beta} + c_2\,;$$

$$U(z, \tau) = c_3\,[(t+\beta)]^{-(\nu+1)/2} \exp\left[-\frac{r^2}{4(t+\beta)}\right] T(r, t)$$

transforms the equation $T'_t = T''_{rr} + \nu r^{-1} T'_r$ to an equation of the same form. For $\nu = 0$ this transformation was found by Appel and subsequently used by Huber in solution of the first boundary-value problem in the indicated region. To construct the Green's function of the first boundary-value problem in the region $[0, R+vt]$, we consider a more general problem of the form

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\,, \quad 0 \leq r < R + vt\,, \quad t > 0 \quad (R \geq 0)\,;$$

$$T(r, 0) = f(r)\,, \quad 0 \leq r \leq R\,; \quad T(r, t)\big|_{r=R+vt} = 0\,, \quad t > 0\,;$$

$$|T(r, t)| < +\infty\,, \quad r \geq 0\,, \quad t \geq 0\,.$$

Let $\tau = 0$ for $t = 0$, and $z = 0$ for $r = 0$ and $z = 1$ for $r = R + vt$. We obtain $\beta = R/v$ and $\gamma = 1/v$. Setting $\alpha = c_1 = 0$ and $c_3 = v^{-1}$, we arrive, in the coordinate system (z, τ), at a classical problem whose solution is found by the method of integral transformations developed in [20, 21]. Finally we find

$$T(r,t) = 2R^{-1}(R+vt)^{-1}\exp\left[-\frac{vr^2}{4(R+vt)}\right]\sum_{n=1}^{\infty}\frac{J_0(\mu_n r/(R+vt))}{J_1^2(\mu_n)}\times$$

$$\times\exp\left[-\frac{\mu_n^2 t}{R(R+vt)}\right]\int_0^1 \xi f(\xi)\exp\left(\frac{v}{4R}\xi^2\right)J_0(\mu_n\xi)\,d\xi\,,$$

where $\mu_n > 0$ are the roots of $J_0(\mu) = 0$. If $f(r) = 2(\pi r)^{-1}\delta(r - r')$, from the above solution it is easy to find the Green's function of the first boundary-value problem for an unbounded continuous cylinder with a uniformly expanding lateral surface. It is also of interest to consider the second and third boundary-value problems and a hollow cylinder $r \in [R_0, R_1 + vt]$, $t \ge 0$ $(R_0 > 0, R_1 > 0)$ and the more complicated case $r \in [R_1 + v_1 t, R_2 + v_2 t]$, $t \ge 0$.

The method of thermal potentials can efficiently be used in finding the analytical solutions of comparatively new problems of heat conduction with an integral boundary condition; these problems are encountered in modeling a number of environmental and biological processes and those of plasma physics and thermomechanics [227]. Here we can obtain results that are of fundamental interest. For example, a solution of the problem

$$\frac{\partial T}{\partial t} = a\frac{\partial^2 T}{\partial x^2}\,,\quad x > 2\sqrt{a}\,y(t)\,,\quad t > 0\,; \tag{38}$$

$$T(x,0) = 0\,,\quad x \ge 0\,;\quad |T(x,t)| < +\infty\,,\quad x \ge 0\,,\quad t \ge 0\,; \tag{39}$$

$$\int_{2\sqrt{a}y(t)}^{\infty} T(x,t)\,dx = 2\sqrt{a}\,y(t)\,, \tag{40}$$

where $y(t)$ is the known time function, should be written in the form of the thermal potential

$$T(x,t) = \int_0^t \frac{\Psi(\tau)}{\sqrt{t-\tau}}\exp\left\{-\frac{[x - 2\sqrt{a}\,y(\tau)]^2}{4a(t-\tau)}\right\}d\tau\,,$$

whose unknown density $\Psi(t)$ is found from the boundary condition (40), which leads to the equation

$$\int_0^1 \Psi(zt)\left[1 - \Phi\left(\frac{y(t) - y(zt)}{\sqrt{t(1-z)}}\right)\right]dz = 2y(t)/\sqrt{\pi t}\,. \tag{41}$$

Let $y(t) = \beta t$ in (38). The operational solution of the integral equation (41) will have the form

$$\Psi(t) = (2\beta/\sqrt{\pi})\,[1 + \beta^2 t + (\beta^2 t + 1/2)\,\Phi(\beta\sqrt{t}) + (\beta/\sqrt{\pi})\sqrt{t}\exp(-\beta^2 t)]\,.$$

For $y(t) = \beta\sqrt{t}$, Eq. (41) has the solution

$$\Psi(t) = \frac{\beta}{\sqrt{\pi\gamma t}}\,;\quad \gamma = [1 + \Phi(\beta)]\,[1 - \sqrt{\pi}\,\beta\exp(\beta^2)\,\Phi^*(\beta)]\,,$$

where $\Phi^*(\beta) = 1 - \Phi(\beta)$; $\Phi(\beta)$ is the Laplace function. Similarly we can also study the remaining cases.

3. Method of Generalized Series. Regions $[0, \gamma\sqrt{2at}]$, $t \geq 0$, and $[\gamma\sqrt{2at}, \infty)$, $t \geq 0$

Systematic study of nonstationary heat conduction in the indicated regions began, apparently, in the 1950s from finding an exact solution of the first boundary-value problem in a semibounded region for a constant temperature at a moving boundary. Subsequently, consideration was given to a bounded region with the following boundary functions: constant, power, and expandable into Maclaurin series (references in [119]). To find the solutions, use was made of implicit techniques depending on the assignment of the functions. The method of [228] generalized different approaches and contributed to further development of the indicated problem. Its main idea lies in the use of series of a generalized form and of special functional transformations which eventually lead to analytical solutions in new functional forms that include all particular cases studied earlier. Thus, for the first boundary-value problem

$$\partial T/\partial t = a\partial^2 T/\partial x^2 , \quad 0 < x < \gamma \sqrt{2at} , \quad t > 0 ; \tag{42}$$

$$T(x, t)\big|_{x=0} = \varphi_1(t) = \sum_{k=-\infty}^{k=+\infty} b_{k/n} t^{k/n} , \quad t > 0 ; \quad T(x, t)\big|_{x=\gamma\sqrt{2at}} = \varphi_2(t) = \sum_{k=-\infty}^{k=+\infty} c_{k/m} t^{k/m} , \quad t > 0 , \tag{43}$$

where n and m are arbitrary real numbers, with the use of successive transformations

$$\xi = \frac{ix}{\sqrt{2at}} \quad (i = \sqrt{-1}) ; \quad T(x, t) \equiv U(\xi, t) = \exp(\xi^2/4)\, W(\xi, t) \tag{44}$$

problem (42)–(43) is transformed relative to the function $W(\xi, t)$:

$$W(\xi, t) = \sum_{k=-\infty}^{k=+\infty} N_{k/n}(\xi)\, t^{k/n} + M_{k/m}(\xi)\, t^{k/m} ,$$

the unknown coefficients being found from satisfying all the conditions of the problem transformed. The final result will be written as

$$T(x, t) = \exp\left(-\frac{x^2}{8at}\right) \sum_{k=-\infty}^{k=+\infty} b_{k/n} \times$$

$$\times \frac{D_{-2k/n-1}(\gamma)\, D_{2k/n}\left(\frac{ix}{\sqrt{2at}}\right) - D_{2k/n}(i\gamma)\, D_{-2k/n-1}\left(\frac{x}{\sqrt{2at}}\right)}{D_{-2k/n-1}(\gamma)\, D_{2k/n}(0) - D_{2k/n}(i\gamma)\, D_{-2k/n-1}(0)}\, t^{k/n} + \exp\left(\frac{\gamma^2}{4}\right) c_{k/m} \times$$

$$\times \frac{D_{2k/m}(0)\, D_{-2k/m-1}\left(\frac{x}{\sqrt{2at}}\right) - D_{-2k/m-1}(0)\, D_{2k/m}\left(\frac{ix}{\sqrt{2at}}\right)}{D_{-2k/m-1}(\gamma)\, D_{2k/m}(0) - D_{2k/m}(i\gamma)\, D_{-2k/m-1}(0)}\, t^{k/m} , \tag{45}$$

where $D_{p/q}(z)$ is the function of a parabolic cylinder. In the particular cases $\varphi_1(t) = b_0$ and $\varphi_2(t) = c_0$, expression (45) yields

$$T(x,t) = b_0 - (b_0 - c_0)\,\Phi\left(\frac{x}{2\sqrt{at}}\right)\Phi^{-1}(\gamma/\sqrt{2})\,.$$

If $\varphi_i(t) = A_i t^{n/2}$ (n = 0, 1, 2, ...) expression (45) is reduced to the form

$$T(x,t) = \sum_{m=1}^{2}\left\{\frac{(-1)^m A_1 n!\, i^{2n}\operatorname{erfc}[(-1)^m\gamma/\sqrt{2}\,] + (-1)^{m-1}A_2/4^n}{i^{2n}\operatorname{erfc}(\gamma/\sqrt{2}\,) + i^{2n}\operatorname{erfc}(-\gamma\sqrt{2}\,)}\;\Phi^*\left[\frac{(-1)^{m-1}x}{2\sqrt{at}}\right]\right\}(4t)^n\,.$$

Similarly we can also consider the second and third boundary-value problems both in a bounded region and a semibounded one and also a system of two media separated by a moving boundary, with boundary conditions of the fourth kind. For the particular values of the boundary functions, the method allows different (equivalent) forms of solutions, which is of importance for applications. The method includes as a particular case a series of self-similar solutions of the equation $T'_t = a(T''_{xx} + \frac{m}{x}T'_x)$ (m = 0, 1, 2) in the form $T(z, t)$ = $t^p\varphi(z)$, $z = x^2/(at)$, where, at the moving boundary $\beta\sqrt{t}$, boundary conditions of the first or second kind are assigned in the form

$$T(x,t)\big|_{x=\beta\sqrt{t}} = t^p\,\varphi(\beta^2/a)\,,\quad t>0\,;\quad \partial T/\partial x\big|_{x=\beta\sqrt{t}} = (2\beta/a)\,\varphi'(\beta^2/a)\,t^{p-1/2}\,,$$

where p is a real number.

Further development of the indicated approach is the passage to a region of the form $x \in [\gamma_1\sqrt{2at}, \gamma_2\sqrt{2at}]$, $t \ge 0$, and to the construction of the Green's functions of the corresponding boundary-value problems in regions with a boundary moving by a parabolic law and a formal extension of the method to the Stefan problems. Of special interest is the development of the method of the Green's functions, taking into account the informativeness of the latter. Here we can propose another method if the problem for $G(x, t, x', \tau)$ is formulated in a form equivalent to (10)–(12), namely,

$$\begin{gathered}\partial G/\partial t = a\partial^2 G/\partial x^2 + \delta(x-x')\,\delta(t-\tau)\,,\quad x>\gamma\sqrt{t}\,,\quad t>0\,;\\ G(x,t,x',\tau)\big|_{t=0} = 0\,,\quad x>0\,;\quad |G(x,t,x',\tau)|_{<+\infty}\,,\quad x\ge 0\,,\quad t\ge 0\,.\end{gathered}\tag{46}$$

To (46) we must add boundary conditions of the form (12). Thus, in the case of the first boundary-value problem the transformations

$$z = \frac{x}{\sqrt{2at}}\,,\quad t' = (1/2)\ln t\,;\quad G(x,t,x',\tau) \equiv G_1(z,t',x',\tau)$$

in combination with the exponential Fourier transformation with respect to the variable $t' \in (-\infty, +\infty)$

$$\overline{G}_1(z,\xi,x',\tau) = (1/\sqrt{2\pi}\,)\int_{-\infty}^{+\infty} G_1(z,t',x',\tau)\exp(-i\xi t')\,dt'$$

reduce the initial problem to the form [229]

$$d^2\overline{G}_1/dz^2 + zd\overline{G}_1/dz - i\xi\overline{G}_1 = -\frac{1}{2\sqrt{\pi a\tau}}\,\tau^{-i\xi/2}\,\delta(z-z_0)\,,\quad z>\gamma/\sqrt{2a}\,,$$

$$\overline{G}_1|_{z=\gamma\sqrt{2a}} = \overline{G}_1|_{z=\infty} = 0 \quad (z_0 = x'/\sqrt{2a\tau}\,)\,, \tag{47}$$

which allows the passage to the following equivalent problem:

$$d^2\overline{G}_1/dz^2 + zd\overline{G}_1/dz - i\xi\overline{G}_1 = 0\,, \quad z > \gamma/\sqrt{2a}\,, \quad \overline{G}_1|_{z=\gamma/\sqrt{2a}} = \overline{G}_1|_{z=\infty} = 0\,; \tag{48}$$

$$\overline{G}_1|_{z=z_0-0} = \overline{G}_1|_{z=z_0+0}\,; \quad \left.\frac{d\overline{G}_1}{dz}\right|_{z=z_0-0} - \left.\frac{d\overline{G}_1}{dz}\right|_{z=z_0+0} = \frac{1}{2\sqrt{a\pi}}\tau^{-(1/2)(1+i\xi)}\,. \tag{49}$$

Condition (49) follows from (47) if we integrate both sides of the equation with respect to $z \in [z_0 - \varepsilon,\ z_0 + \varepsilon]$ and pass to the limit when $\varepsilon \to 0$. To solve problem (48)–(49), we must use the theory of functions of a parabolic cylinder that satisfy (48) and then return to the space of inverse transforms according to the inversion formula for the Fourier transformation. Passage to the equivalent problem is a very effective approach that can also be used for other laws of motion of a boundary. However, investigations for the regions indicated in this part of the review are first of all required. Here, just as in other cases, one should be attentive in formulating boundary conditions for the corresponding Green's function, especially for the second and third boundary-value problems (in the form of (12)). The formal passage to relations (16)–(17) was the reason why a number of works on this problem were incorrect.

4. Method of a Generalized Integral Transformation. Region $[0, y(t)]$, $t \ge 0$, $y(0) \ge 0$. Stefan Problem

In [44, 201, 230], it is proposed to generalize the method of integral transformations to the noncylindrical region $x \in [0, y(t)]$, $t \ge 0$, including the case $y(t) = l = \text{const}$, too. Especially efficient was the formal application of the method to solving the Stefan problem and a more general Stefan problem – boundary-value problems for the heat-conduction equation with a free boundary. The features of the method are the possibility of considering nondegenerate regions with $y(0) = y_0 > 0$ and a rather wide class of functions in the inhomogeneities of the initial formulation of the problem. As a result, the method leads to the analytical solutions in a new integral form for cases not subject earlier to the study by other approaches. To solve the equation

$$\frac{\partial T}{\partial t} = a\left(\frac{\partial^2 T}{\partial x^2} + \frac{1-2m}{x}\frac{\partial T}{\partial x}\right) + f(x, t) \tag{50}$$

for $m = -1/2$, $0(0 \le x < y(t),\ t > 0)$; $1/2(0 < x < y(t),\ t > 0)$, we introduce the generalized integral transformation

$$\overline{T}(p, t) \int_0^{y(t)} x^{1-m}\, T(x, t)\, J_m(x\sqrt{p}\,)\, dx\,,$$

$$p = \beta + i\omega\,; \quad \mathrm{Re}\, p \le \sigma < 0\,; \quad -\pi/4 < \arg\sqrt{p} < \pi/4\,, \tag{51}$$

an inversion formula for which is written in the form

$$T(x, t) = \frac{2}{y^2(t)} \sum_{n=1}^{\infty} \frac{a_n(t)\exp[-(\sqrt{a}\,\mu_n/y^2(t))\,t]}{J_{m-1}^2(\mu_n)}\, x^m J_m\left(\frac{\mu_n x}{y(t)}\right). \tag{52}$$

Here $\mu_n > 0$ are the roots of the equation; $J_m(\mu) = 0$ and a_n are the unknown coefficients to be found. In (51)–(52), it suffices to consider the cases $m = 0$ and $m = 1/2$ since the case $m = -1/2$ is reduced to $m =$

1/2. Equation (50) and expression (52) are converted to the space of transforms (51), and then by way of contour integration using the Cauchy theorem we successively write a_n in the form

$$a_n(t) = \int_0^{y_0} x^{1-m} T(x, 0) J_m\left(\frac{\mu_n x}{y(t)}\right) dx + \int_0^t \exp\left[\left(\frac{\sqrt{a}\,\mu_n}{y(t)}\right)^2 \tau\right] \times$$

$$\times T(y(\tau), \tau)\, y^{1-m}(\tau) \left[\frac{dy}{d\tau} J_m\left(\frac{\mu_n y(\tau)}{y(t)}\right) - \frac{a\mu_n}{y(t)} J_{m-1}\left(\frac{\mu_n y(\tau)}{y(t)}\right)\right] d\tau +$$

$$+ a \int_0^t y^{1-m}(\tau) J_m\left(\frac{\mu_n y(\tau)}{y(t)}\right) \frac{\partial T(y(\tau), \tau)}{\partial x} \exp\left[\left(\frac{\sqrt{a}\,\mu_n}{y(t)}\right)^2 \tau\right] d\tau +$$

$$+ a \left(\frac{\mu_n}{y(t)}\right)^m \delta_m \int_0^t \exp\left[\left(\frac{\sqrt{a}\,\mu_n}{y(t)}\right)^2 \tau\right] T(0, \tau)\, d\tau + \int_0^t \int_0^{y(\tau)} \exp\left[\left(\frac{\sqrt{a}\,\mu_n}{y(t)}\right)^2 \tau\right] \times$$

$$\times x^{1-m} f(x, \tau) J_m\left(\frac{\mu_n x}{y(t)}\right) d\tau dx\,, \tag{53}$$

where $\delta_m = 2^{1-m}/\Gamma(m)$, $m > 0$ and $\delta_m = 0$, $m \le 0$. In the process of determination of the coefficients $a_n(t)$, we find an integral equation that relates the boundary functions of the problem to the law of motion of a boundary (for Re $p < 0$):

$$a \int_0^{\infty} \frac{\partial T(y(\tau), \tau)}{\partial x} y^{1-m}(\tau) J_m(y(\tau)\sqrt{p}) \exp(ap\tau)\, d\tau + \int_0^{y_0} x^{1-m} T(x, 0) J_m \times$$

$$\times (x\sqrt{p})\, dx + ap^{m/2} \delta_m \int_0^{\infty} T(0, \tau) \exp(ap\tau)\, d\tau = \int_0^{\infty} T(y(\tau), \tau)\, y^{1-m}(\tau) \times$$

$$\times \left[a\sqrt{p}\, J_{m-1}(y(\tau)\sqrt{p}) - \frac{dy(\tau)}{d\tau} J_m(y(\tau)\sqrt{p})\right] \exp(ap\tau)\, d\tau -$$

$$- \int_0^{\infty} \int_0^{y(t)} f(x, \tau)\, x^{1-m} J_m(x\sqrt{p}) \exp(ap\tau)\, dx d\tau\,. \tag{54}$$

Expressions (52)–(54) represent all the relations required for consideration of a number of boundary-value problems for Eq. (50) with a free boundary that can be described analytically by the analytical solutions (52). From (53) it is obvious that we are dealing with the first boundary-value problem when the normal derivative is additionally assigned at a moving boundary; the unknown law of motion of the boundary is found from (54). The method also allows a generalization to other types of boundary conditions with the corresponding selection of the kernels of the transforms in (51) and to the regions $[y(t), \infty)$, $t \ge 0$ (here use can be made of the kernels exp $(-x\sqrt{p})$ in a Cartesian system and $rK_0(r\sqrt{p})$ for cylindrical symmetry ($K_0(z)$ is the MacDonald function)) and in this sense represents a further direction in investigations.

In the particular cases of boundary functions in (53), relations (52)–(54) yield interesting new results. For example, a solution of the simplest Stefan problem

$$\partial T/\partial t = a\partial^2 T/\partial x^2 \,, \quad 0 < x < y\,(t)\,, \quad t > 0\,, \quad y\,(0) = 0\,,$$
$$T\,(0, t) = \varphi_0 = \text{const} < 0\,, \quad t > 0\,; \quad T\,(y\,(t), t) = 0\,, \quad t > 0\,; \quad \partial T/\partial x\,|_{x=y(t)} = A dy/dt\,, \quad t > 0\,, \tag{55}$$

is well known and has the form [139]

$$y\,(t) = \beta\,\sqrt{t}\,; \quad -2\varphi_0/(A\,\sqrt{\pi a}\,) = \beta \exp\,(\beta^2/4a)\;\Phi\,(\beta/(2\,\sqrt{a}\,)\,; \tag{56}$$

$$T\,(x, t) = \varphi_0 + (A\beta\,\sqrt{\pi a}/2)\exp\,(\beta^2/4a)\Phi\left(\frac{x}{2\,\sqrt{at}}\right). \tag{57}$$

The approach (52)–(54) yields

$$T\,(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\beta\,\sqrt{t}}\,; \quad b_n = \frac{A\beta^2}{n\pi}\sum_{k=1}^{\infty}\frac{(-1)^k\,(an^2\pi^2/\beta^2)^k}{k\,!}\times\sum_{m=k}^{\infty}\frac{\beta^{2m}\,m\,!}{a^m\,(2m+1)\,!}\,; \tag{58}$$

$$y\,(t) = \beta\,\sqrt{t}\,; \quad -2\varphi_0/(A\,\sqrt{\pi a}\,) = \beta\exp\,(\beta^2/4a)\;\Phi\,(\beta/2\,\sqrt{a}\,)\,. \tag{59}$$

In [44], the equivalence of expressions (57)–(58) is proved, For the case of cylindrical symmetry with the known law of motion of a boundary, a problem of the form

$$\frac{\partial T}{\partial t} = a\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right), \quad 0 \le r < \beta\,\sqrt{t}\,, \quad t > 0\,; \tag{60}$$

$$(\partial T/\partial r)_{r=\beta\sqrt{t}} = q\,\sqrt{t}\,, \quad t > 0 \quad (q = \text{const}\,)\,; \quad |\,T\,(r, t)\,| < +\infty\,, \quad r \ge 0\,, \quad t > 0\,, \tag{61}$$

has the solution

$$T\,(r, t) = (aq/2\beta)\;t\,(4 + r^2/at)\,. \tag{62}$$

From (62) we find the boundary condition $T(y(t), t) = q(\beta^2 + 4a)t/(2\beta)$. A similar expression is obtained from the integral relation (54), which can clearly be seen. We can also consider more complicated cases different from (61).

5. Method of Differential Series. Region [0, $y(t)$], $t \ge 0$, $y(0) \ge 0$

In a series of works [16, 56, 75, 80] (and references in [119]), consideration is given to another approach to a region with an arbitrarily moving boundary. Its practical use assumes a computation of the derivatives of any order of the expressions of a special form in the common term of a series. The method makes it possible to obtain an analytical solution of the problem for any form of boundary conditions, including the case $h = h(t)$ in the boundary condition of the third kind at the boundaries of the region. The method is especially efficient in the solution of inverse problems for a heat-conduction equation and also of inverse Stefan problems where the temperature (or the heat flux) at a fixed boundary, the initial distribution of the temperature, and its value in a medium are found by the known law of motion of a boundary and Stefan conditions at a moving boundary. It is essential that the physical content of the problem remains constant for the most part, but the initial problem is simplified and in order to solve it one of the above approaches can be applied. It is shown that the solution$T(x, t)$ of the nonhomogeneous equation

$$\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2} + f(x, t)\,, \quad 0 < x < y(t)\,, \quad t > 0\,, \tag{63}$$

can be written in the form

$$T(x, t) = T(y(t), t) + \sum_{n=1}^{\infty} \frac{1}{a^n (2n)!} \frac{\partial^{n-1}}{\partial t^{n-1}} \left\{ [y(t) - x]^{2n} \frac{d}{dt} T(y(t), t) \right\} -$$

$$- \sum_{n=0}^{\infty} \frac{1}{a^n (2n+1)!} \frac{\partial^n}{\partial t^n} \left\{ [y(t) - x]^{2n+1} \frac{\partial T(y(t), t)}{\partial x} \right\} -$$

$$- \sum_{n=0}^{\infty} \frac{1}{a^{n+1} (2n+1)!} \frac{\partial^n}{\partial t^n} \int_x^{y(t)} (\xi - x)^{2n+1} f(\xi, t)\, d\xi\,. \tag{64}$$

It represents the basic formula in the solution of boundary-value problems for Eq. (63). Relation (64) can be called a generalized Cauchy series for Eq. (63) where the Cauchy conditions are assigned on the analytical (infinitely differentiable) curve $x = y(t)$. Let, for Eq. (63) when $y(0) = y_0 > 0$, the following Stefan boundary conditions be assigned:

$$T(x, 0) = \Phi_0(x)\,, \quad 0 < x < y_0\,; \quad T(y(t), t) = T_{\mathrm{cr}}\,, \quad t > 0\,; \tag{65}$$

$$\partial T/\partial x\,|_{x=y(t)} = A dy/dt\,, \quad t > 0\,, \tag{66}$$

where $A = \pm\,\alpha_n(\rho/\lambda)$, α_n is the heat of transformation, ρ is the density, and λ is the thermal conductivity; the sign (+) corresponds to the coding and the sign (–) to the heating of the medium of surface $x = 0$. On this surface we can assign one of the three boundary conditions (of the first, second, or third kind)

$$(\beta_1 \partial T/\partial x + \beta_2 T)_{x=0} = \beta_3\, \varphi(t)\,, \quad t > 0\,. \tag{67}$$

Considering the law of motion of a boundary and the Cauchy conditions at a moving boundary to be assigned, we write $T(x, t)$ from (64) (for $f(x, t) = 0$):

$$T(x, t) = T_{\mathrm{cr}} - A \sum_{n=1}^{\infty} \frac{1}{a^n (2n)!} \frac{\partial^n}{\partial t^n} \left\{ [y(t) - x]^{2n} \right\}, \tag{68}$$

whence it is easy to write the initial distribution of the temperature and (for example) its value for $x = 0$:

$$\Phi_0(x) = T_{\mathrm{cr}} - A \sum_{n=1}^{\infty} \frac{1}{a^n (2n)!} \left\{ \frac{\partial^n}{\partial t^n} [y(t) - x]^{2n} \right\}_{t=0}\,; \qquad \varphi(t) = T_{\mathrm{cr}} - A \sum_{n=1}^{\infty} \frac{1}{a^n (2n)!} [y^{2n}(t)]^{(n)}\,. \tag{69}$$

If $y(0) = 0$, the considered region is degenerate, i.e., at the initial instant it is reduced to the point $x = 0$ and has initial temperature equal to zero ($y_0 = 0$, $\Phi_0(x) = 0$). For this case, the method yields efficient relations for finding the unknown law of motion of a boundary, which means the actual solution of primal Stefan problems and more general ones for a heat-condition equation with a free boundary. Thus, for the above Cauchy conditions and the first-kind boundary conditions in (67) the method yields (for $f(x, T) = 0$) the following relation to find $y(t)$:

$$\sum_{n=1}^{\infty} [y^{2n}(\tau)]^{(n)}/(2n)! = [T_{cr} - \varphi(\tau)]/Aa \quad (\tau = at) . \tag{70}$$

In the space of the Laplace transforms $\int_0^\infty \ldots \exp(-pt)dt$, Re $p \geq \beta > 0$, $|\arg \sqrt{p}| < \pi/4$, expression (70) is transformed to the integral equation

$$\int_0^\infty \exp(-p\tau) \cosh[y(\tau)\sqrt{p}\,]\, d\tau = \frac{T_{cr} + aA}{aAp} - \frac{\overline{\varphi}(p)}{aA} , \tag{71}$$

which is solved exactly by the known approaches for a number of particular values of the boundary function $\varphi(t)$.

In more complicated cases, (71) makes it possible to study the asymptotic character of motion of a boundary for larger times (references in [119]). Similar relations can be obtained with boundary conditions of the second and third kind in (67). In the case of cylindrical and spherical symmetry for the Stefan problem of the form

$$\begin{gathered} \frac{\partial T}{\partial t} = a \frac{1}{r^m} \frac{\partial}{\partial r}\left(r^m \frac{\partial T}{\partial r}\right), \quad 0 \leq r < y(t) , \quad t > 0 ; \\ T(y(t), t) = T_{cr}\,(t > 0) ; \quad (\partial T/\partial r)_{r=y(t)} = A dy/dt + \varphi(t) , \quad t > 0 ; \\ \partial T/\partial r\,|_{r=0} = 0 , \quad t > 0 \quad (m = 1, 2 ; \quad y(0) = 0) , \end{gathered} \tag{72}$$

the method yields the following relations for $y(t)$:

$$\sum_{n=0}^{\infty} \frac{[y^{(2n+1)}(\tau)]^{(n)}}{[(2n)!!]^2} \left[\frac{dy}{d\tau} + \frac{\varphi(\tau)}{aA}\right] = 0 , \quad m = 1 \quad (\tau = at) ;$$

$$\sum_{n=0}^{\infty} \frac{[y^{2(n+1)}(\tau)]^{(n)}}{(2n+1)!} \left[\frac{dy}{d\tau} + \frac{\varphi(\tau)}{aA}\right] = 0 , \quad m = 2 ,$$

which can be studied using the known approaches of computational mathematics. For a number of particular laws of motion of a boundary $y(t)$, the series in (64) can be transformed to power ones or summed up. Thus, for a variable velocity of motion of a boundary $v(t) = bt$ and constant temperature φ_0 and heat flux q_0 at a moving boundary, expression (64) (for $f = 0$) is reduced to the form [154]

$$T(x,t) = \varphi_0 + q_0 \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!\, b^n (x - bt^2/2)^{3n+1}}{a^{2n} (3n)!\, n!\, 2^n},$$

whence it is easy to obtain the temperature regime at a fixed boundary. For $y(\tau) = \sqrt{\beta\tau + h^2}$ ($\tau = at$) and condition (66), the series (68) (when $T_{cr} = 0$) yields

$$T(x,t) = -aA\sqrt{\pi B}\exp(B/4)\left[\Phi(\sqrt{B}/2) - \Phi\left(\frac{x}{2}\sqrt{\frac{B}{B\tau + h^2}}\right)\right],$$

whence the functions (69) corresponding to the assigned law of motion of a boundary are found. Similarly, we can also consider other laws of motion of a boundary, in particular, $y(\tau) = \sqrt{A\tau^2 + B\tau + C}$, including the spherically symmetric layer $R_0 < r < R(\tau)$ for the equation in (72) (references in [119]).

One problem of the presented approach is the establishment of the rate of convergence of the series in (64).

6. Method of Functional Transformations. New Laws of Motion

G. A. Grinberg proposed efficient functional transformations for the conversion of regions with moving boundaries to regions with fixed boundaries that considerably increase the number of concrete laws of motion of a boundary, for which it is possible to obtain an exact analytical solution of the problem (references in [119]). The cases of variation in the region with retention of similarity $x_i \in [0, y(t)]$, $t \geq 0$ ($i = 1, 2, 3$); without retention of similarity $x_i \in [0, y_i(t)]$, $t \geq 0$ ($i = 1, 2, 3$); the region $x_i \in [y_i^{(0)}(t), y_i^{(1)}(t)]$, $[y_i(t), \infty)$, $t \geq 0$ ($i = 1, 2, 3$) have successively been considered. All the $y_i(t)$ are continuously differentiable functions of second order inclusive.

For expanding or contracting regions with retention of similarity, the equation

$$\partial T/\partial t = a\Delta T(M,t) + f(M,t), \quad M = M(x_1, x_2, x_3), \quad 0 < x_i < y(t), \quad t > 0, \tag{73}$$

is successively transformed to

$$\xi_i = x_i/y(t), \quad T(M,t) \equiv U(P,t), \quad P = P(\xi_1, \xi_2, \xi_3);$$

$$U(P,t) = [y(t)]^{-n/2}\exp\left[-\frac{1}{4a}y(t)\,y'(t)\sum_{i=1}^{3}\xi_i^2\right]W(P,t). \tag{74}$$

Here $n = 3$ for a spatial problem, $n = 2$ for a planar problem, and $n = 1$ for a one-dimensional problem. Equation (73) becomes as follows:

$$y^2(t)\frac{\partial W}{\partial t} = a\Delta W(P,t) + \frac{1}{4a}y^3(t)\,y''(t)\left(\sum_{i=1}^{3}\xi_i^2\right)W + F(P,t), \quad 0 < \xi_i < 1, \quad t > 0, \tag{75}$$

where $F(P, t)$ is the new (unknown) function. If the "new" time $\tau = \int_0^t y^{-2}(t')dt$ is introduced, taking into account that $d\tau/dt > 0$ and τ increase with t, (75) will take the form

$$\frac{\partial W}{\partial \tau} = a\Delta W(P, \tau) + \frac{1}{4a} y^3(\tau)\, y''(\tau) \left(\sum_{i=1}^{3} \xi_i^2 \right) W + F(P, \tau)\,, \quad 0 < \xi_i < 1\,, \quad \tau > 0\,, \tag{76}$$

and in this form differs from the initial equation (73) just by the presence of the second term on the right-hand side (but in the region with fixed boundaries). In the particular case where $y''(t) = 0$, i.e., $y(t) = At + B$, the difference of these equations disappears completely. The homogeneous equation (75) allows the separation of variables (in Cartesian, cylindrical, and spherical coordinates), and following this direction we can find the solution of the first boundary-value problem in the known functions for uniformly expanding or contracting regions that take, in (75), the shape of an unbounded plate ($n = 1$), a rectangle ($n = 2$), a rectangular parallelepiped ($n = 3$), a cylinder of finite length, a sphere or a spherical layer, etc. With regard to the second and third boundary-value problems, here, in the initial region, it is more expedient to construct the Green's functions since the transform (74) converts boundary conditions of the second and third kind to a boundary condition of the third kind with a time-variable coefficient of heat exchange. The exact solutions of this class of problems are very cumbersome [119]. In constructing the Green's function in the spatial region, use can be made of the method of the product of solutions of one-dimensional problems. In the case where $y^3(t)y''(t) = -\alpha = \text{const} \neq 0$ in (95), i.e., $y(t) = \sqrt{(At + B)^2 - \alpha/A^2}$ (A and B are const), Eq. (75) also allows the separation of variables; precisely the same problems as for a uniform law of motion of a boundary can be solved for it. We should note the especially important case $y(t) = \sqrt{Mt + N}$ for $y^3(t)y''(t) = -(M^2/4)$ where the solution of the initial problem with boundary conditions of any kind is possible. These solutions can have the form of functional constructions different from (45).

For the regions that change with time without retention of similarity, Eq. (73) is transformed using the relations

$$\xi_i = x_i/y_i(t)\,, \quad T(M, t) \equiv U(P, t)\,, \quad P = P(\xi_1, \xi_2, \xi_3)\,;$$

$$U(P, t) = \left[\prod_{i=1}^{3} y_i(t) \right]^{-1/2} \exp\left[-\frac{1}{4a} \sum_{i=1}^{3} y_i(t)\, y_i'(t)\, \xi_i^2 \right] W(P, t) \tag{77}$$

and becomes as follows:

$$\frac{\partial W}{\partial t} = a \sum_{i=1}^{3} \frac{1}{y_i^2(t)} \left[\frac{\partial^2 W}{\partial \xi_i^2} + \frac{1}{4a^2} \xi_i^2\, y_i''(t)\, y_i^3(t)\, W \right] + F(P, t)\,. \tag{78}$$

If $y_i''(t)y_i^3(t) = -\alpha_i$, i.e., $y_i(t) = \sqrt{(A_i t + B_i)^2 - \alpha_i/A_i^2}$, then (78) (for $F = 0$) allows the separation of variables, and the first boundary-value problem can be solved exactly. Here we should also single out the case $y_i(t) = \sqrt{M_i t + N_i}$, which makes it possible to obtain the solution of the initial problem with boundary conditions of any order at moving boundaries.

In the cylindrical coordinates $M(x_1, \varphi, x_2)$, the equation

$$\partial T/\partial t = a\Delta T(M, t) + f(M, t)\,, \quad 0 \leq x_1 < y_1(t)\,, \quad 0 < x_2 < y_2(t)\,, \quad 0 \leq \varphi \leq 2\pi\,, \quad t > 0\,, \tag{79}$$

by the transformations

$$\xi_i = x_i/y_i(t)\,, \quad T(M, t) \equiv U(P, t)\,, \quad P = P(\xi_1, \varphi, \xi_2)\,,$$

$$U(P,t)=\frac{1}{y_1(t)\sqrt{y_2(t)}}\exp\left[-\frac{1}{4a}\sum_{i=1}^{2}\xi_i^2\,y_i(t)\,y_i'(t)\right]W(P,t) \tag{80}$$

is reduced to the form

$$\frac{1}{a}\frac{\partial W}{\partial t}=\sum_{i=1}^{2}\frac{1}{y_i^2(t)}\left[\frac{\partial^2 W}{\partial \xi_i^2}+\frac{\xi_i^2}{4a^2}y_i^3(t)\,y_i''(t)\,W\right]+\frac{1}{\xi_1}\frac{\partial W}{\partial \xi_1}+\frac{1}{\xi_1^2}\times$$

$$\times\frac{\partial^2 W}{\partial \varphi^2}+F(P,t)\,,\quad 0\le\xi_1<1\,,\quad 0<\xi_2<1\,,\quad 0\le\varphi\le 2\pi\,,\quad t>0\,. \tag{81}$$

If, just as above, $y_i^3(t)y_i''(t) = -\alpha_i$, i.e., $y_i(t) = \sqrt{(A_it+B_i)^2-\alpha_i/A_i^2}$ and in the particular cases of this general dependence $y_i(t) = A_it+B_i$ and $y_i(t) = \sqrt{M_it+N_i}$, then (81) can be solved by the known approaches.

Let now (73) be assigned in the region $x_i \in [y_i^{(0)}(t), y_i^{(1)}(t)]$, $t\ge 0$. Using the relations

$$\xi_i=[x_i-y_i^{(0)}(t)]/\eta_i(t)\,;\quad \eta_i(t)=y_i^{(1)}(t)-y_i^{(0)}(t)\,;\quad T(M,t)\equiv U(P,t)\,;$$

$$U(P,t)=W(P,t)\prod_{i=1}^{3}\eta_i^{-1/2}(t)\exp\left[-\frac{1}{4a}\left(\eta_i\,\eta_i'\,\xi_i^2+2\eta_i\,y_i^{(0)}\,\xi_i+\int_0^t\left[y_i^{(0)'}(\tau)\right]^2 d\tau\right)\right] \tag{82}$$

we transform Eq. (73) to the form

$$\frac{\partial W}{\partial t}=a\sum_{i=1}^{3}\frac{1}{\eta_i^2}\left[\frac{\partial^2 W}{\partial \xi_i^2}+\frac{1}{4a^2}\left(\eta_i^3\,\eta_i''\,\xi_i^2+2\eta_i^3\,y_i^{(0)''}\,\xi_i\right)W\right]+F(P,t)\,,\quad 0<\xi_i<1\,,\quad t>0\,, \tag{83}$$

and in this form it allows an exact solution in the known functions if for all $t>0$ the conditions $\eta_i^3\eta_i''$ = const and $y_i^3y_i^{(0)''}$ = const are simultaneously fulfilled. This means:

$$\text{if}\quad \eta_i=\text{const}\,,\quad y_i^{(0)}=A_it^2+B_it+C_i\,; \tag{84}$$

$$\text{if}\quad \eta_i=\sqrt{A_it^2+B_it+C_i}\,,\quad y_i^{(0)}=D_i\sqrt{A_it^2+B_it+C_i}+M_it+N_i\,; \tag{85}$$

$$\text{if}\quad \eta_i=A_it+B_i\,,\quad y_i^{(0)}=C_i\,(A_it+B_i)^{-1}+D_it+E_i\,. \tag{86}$$

The particular cases of the motions (84)–(86) are considered above. In the cylindrical coordinates $M(x_1, \varphi, x_2)$ for $x_1 \in [y_1(t), \alpha y_1(t)](\alpha>1)$, $x_2 \in [y_2^{(0)}(t), y_2^{(1)}(t)]$, and $\varphi \in [0, 2\pi]$, $t\ge 0$, we introduce the transformations

$$\xi_1=x_1/y_1(t)\,,\quad \xi_2=[x_2-y_2^{(0)}(t)]/\eta(t)\,,\quad \eta(t)=y_2^{(1)}(t)-y_2^{(0)}(t)\,;\quad T(M,t)\equiv U(P,t)\,,$$

$$P=P(\xi_1,\varphi,\xi_2)\,;\quad U(P,t)=W(P,t)\,y_1^{-1}(t)\,\eta^{-1/2}(t)\times$$

$$\times \exp\left[-\frac{1}{4a}\left(\xi_1^2 y_1 y_1' + \eta\eta'\xi_2^2 + 2\eta\xi_2 y_2^{(0)'} + \int_0^t (y_2^{(0)'}(\tau))^2 d\tau\right)\right]. \tag{87}$$

Equation (73) becomes as follows:

$$\frac{1}{a}\frac{\partial W}{\partial t} = \frac{1}{y_1^2}\left(\frac{\partial^2 W}{\partial \xi_1^2} + \frac{1}{\xi_1}\frac{\partial W}{\partial \xi_1} + \frac{1}{4a^2}\xi_1^2 y_1^3 y_1'' W + \frac{1}{\xi_1^2}\frac{\partial^2 W}{\partial \varphi^2}\right) +$$

$$+ \frac{1}{\eta^2}\left[\frac{\partial^2 W}{\partial \xi_2^2} + \frac{1}{4a^2}\left(\eta^3\eta''\xi_2^2 + 2\eta^3 y_1^{(0)''}\xi_2\right)W\right] + F(P, t),$$

$$1 < \xi_1 < \alpha, \quad 0 < \xi_2 < 1, \quad 0 \le \varphi \le 2\pi, \quad t > 0, \tag{88}$$

and allows an exact solution in the known functions if the conditions $y_1^3 y_1''$ = const, $\eta^3\eta''$ = const, and $\eta^3 y_1^{(0)''}$ = const simultaneously occur; for $y_1(t) = \sqrt{A_1 t^2 + B_1 t + C_2}$, the functions $y_2^{(0)}(t)$ and $y_2^{(1)}(t)$ satisfy equations of motion of the form (84)–(86). Thus, for the initial equation (73) the first boundary-value problem is solved: a) for a parallelepiped one pair of whose parallel sides moves in the direction of its axis by the law (84), the second pair of which moves by the law (85), and the third pair – by the law (86); b) for a bounded hollow cylinder whose lateral surfaces move by the law given above for $y_1(t)$ and $\alpha y_1(t)$ and whose ends move by any of the laws (84)–(86).

With regard to the semibounded region $x_i > y_i(t)$, $t > 0$ (i = 1, 2, 3), for Eq. (73) here we introduce the transformations

$$\xi_i = x_i - y_i(t), \quad T(M, t) \equiv U(P, t), \quad P = P(\xi_1, \xi_2, \xi_3),$$

$$U(P, t) = W(P, t)\exp\left\{-\frac{1}{2a}\left[\sum_{i=1}^{3}\xi_i y_i'(t) + \frac{1}{2}\int_0^t\left(\sum_{i=1}^{3} y_i'^2(\tau)\right)d\tau\right]\right\}, \tag{89}$$

which make it possible to represent (73) as follows:

$$\frac{\partial W}{\partial t} = a\Delta W(P, t) + \frac{1}{2a}\left(\sum_{i=1}^{3} y_i''(t)\,\xi_i\right)W + F(P, t); \quad \xi_i > 0, \quad t > 0. \tag{90}$$

It follows from (90) that when $y_i''(t)$ = 0, i.e., $y_i(t) = A_i t + B_i$, we have the classical case; when $y_i''(t) = \alpha_i \ne 0$ and $y_i(t) = \alpha_i t^2/2 + \beta_i t$ the equation allows the separation of variables. Relations (77) and (89) make it possible to consider a combination of regions with moving boundaries; too: bounded regions in some space variables and semibounded in other derivatives. Here we can also indicate certain particular cases of solvability of spatial problems in the known functions. For example, for

$$\partial T/\partial t = a\Delta T(M, t), \quad x_1 > y_1(t) = \alpha t^2/2 + \beta t + \gamma, \quad 0 < x_i < y_i(t), \quad t > 0 \quad (i = 2, 3),$$

we apply expression (89) with respect to x_1 (i = 1) and pass to (90).

We then separate the variables, setting $W = \mu(\xi_1)\Theta(x_2, x_3, t)\exp(-\lambda t)$. We find

$$\mu'' + \left(\frac{\alpha}{2a^2}\xi_1 - \lambda\right)\mu = 0\,, \quad \xi_1 > 0\,; \quad \partial\Theta/\partial t = a\Delta\Theta\,(x_2, x_3, t)\,, \quad 0 < x_i < y_i\,(t)\,, \quad t > 0\,.$$

The spectral problem for $\mu(\xi_1)$ is considered below; the equation for $\Theta(x_2, x_3, t)$ is transformed using (77)–(78). In particular, when $y_i(t) = \sqrt{A_i t^2 + B_i t + C_i}$ $(i = 2, 3)$, (78) will take the form

$$\frac{1}{a}\frac{\partial W}{\partial t} = \sum_{i=2}^{3} y_i^{-2}\,(t)\left(\frac{\partial^2 W}{\partial \xi_i^2} + \frac{A_i C_i - B_i^2}{4a^2}\xi_i^2 W\right), \quad 0 < \xi_i < 1\,, \quad t > 0\,. \tag{91}$$

In this equation, the variables are separated if we set $W = \varphi_1(t)\varphi_2(\xi_2)\varphi_3(\xi_3)$ and subject the functions φ_i to the conditions

$$\varphi_1'\,(t) = \left[\sum_{i=2}^{3}\frac{\lambda_i}{y_i^2\,(t)}\right]\varphi_1\,(t)\,; \quad \frac{\partial^2\varphi_i}{\partial\xi_i^2} + \left(\frac{A_i C_i - B_i^2}{4a^2}\xi_i^2 + \lambda_i\right)\varphi_i = 0\,,$$

where λ_i $(i = 1, 2)$ are the separation variables. The solutions of these equations are expressed in terms of confluent hypergeometric functions. If $y_2(t) = y_3(t)$ (uniform expansion or contraction), then Eq. (91) is simplified

$$\frac{1}{a}\frac{\partial W}{\partial t} = y_2^{-2}\,(t)\left[\Delta W\,(\xi_2, \xi_3, t) + \frac{A_2 C_2 - B_2^2}{4a^2}(\xi_2^2 + \xi_3^2)\,W\right]$$

and in this form allows the separation of variables not only in Cartesian coordinates but also in polar coordinates. The problem is solved exactly for circular regions, too. In each of the considered cases of motion of a boundary where the transformed equation allows the separation of variables, the solution of the problem can be brought to completion according to the following scheme. In the first step, the corresponding problem in eigenvalues and eigenfunctions is solved; then, based on the solution found, we introduce an integral transformation, its inversion formula, and a transform of the second partial derivative. Thus, the required body of mathematics that makes it possible to write the solution sought is constructed in advance. Following this direction, we can obtain results of interest for the theory of special functions in solving spectral problems for ordinary differential equations of second order. Let us consider one of these characteristic examples of finding the solution of the first boundary-value problem for Eq. (73) in the region $x > y(t)$, $t > 0$, in the one-dimensional case (for $f = 0$). The transformations (89)–(90) $(i = 1)$ for the case $y''(t) = -a^2\omega$ $(\omega > 0)$ yield

$$\frac{1}{a}\frac{\partial W}{\partial t} = \frac{\partial^2 W}{\partial\xi^2} - \frac{1}{2}\omega\xi W\,, \quad \xi > 0\,, \quad t > 0\,; \tag{92}$$

$$W\,(\xi, 0) = 0\,, \quad \xi \ge 0\,; \quad W\,(0, t) = \varphi\,(t)\,, \quad t \ge 0\,; \quad |W\,(\xi, t)| < +\infty\,, \quad \xi \ge 0\,, \quad t \ge 0\,. \tag{93}$$

Equation (92) has particular solutions of the form $W = \Psi(\xi)\exp(-a\omega\gamma^2 t/2)$, where γ^2 is the separation constant. The spectral problem

$$\frac{d^2\Psi}{d\xi^2} + (1/2)\,\omega\,(\gamma^2 - \xi)\,\Psi\,(\xi) = 0\,, \quad \xi > 0\,; \quad \Psi\,(\xi)\,|_{\xi=0} = \Psi\,(\xi)\,|_{\xi=\infty} = 0$$

has the solution

$$\Psi_n(\xi) = (\pi/\sqrt{3})\sqrt{\gamma_n^2 - \xi}\,\{J_{1/3}[(1/3)\sqrt{2\omega}\,(\gamma_n^2 - \xi)^{3/2}] + J_{-1/3}[(1/3)\sqrt{2\omega}\,(\gamma_n^2 - \xi)^{3/2}]\}\,;\ \gamma_n^2 = (9\mu_n^2/2\omega)^{1/3}\,,$$

where $\mu_n > 0$ are roots of the equation $J_{1/3}(\mu) + J_{-1/3}(\mu) = 0$.

The integral transformation and the inversion formula for it have the form

$$\overline{W}(\gamma_n, t) = \int_0^\infty W(\xi, t)\,\Psi_n(\xi)\,d\xi\,;\quad W(\xi, t) = \sum_{n=1}^\infty \frac{\Psi_n(\xi)}{\|\Psi_n\|^2}\,\overline{W}(\gamma_n, t)\,,$$

where

$$\|\Psi_n\|^2 = (1/3)\,\pi^2\,\gamma_n^2\,[J_{-2/3}(\mu_n) - J_{2/3}(\mu_n)]^2\,.$$

Here,

$$\int_0^\infty (\partial^2 W/\partial\xi^2 - (1/2)\,\omega\xi W)\,\Psi_n(\xi)\,d\xi = -\frac{\pi\gamma_n\sqrt{\omega}}{\sqrt{6}}\,\varphi(t)\,[J_{-2/3}(\mu_n) - J_{2/3}(\mu_n)] - (1/2)\,\omega\gamma_n^2\,\overline{W}(\gamma_n, t)\,.$$

The required body of mathematics is constructed. New we write the solution of the sought problem (92)–(93):

$$W(\xi, t) = \sum_{n=1}^\infty \frac{\Psi_n(\xi)}{\|\Psi_n\|^2}\left\{-\frac{\pi a\gamma_n}{\sqrt{3}}\sqrt{\omega/2}\,[J_{-2/3}(\mu_n) - J_{2/3}(\mu_n)]\int_0^t \exp[-(1/2)\,\omega\gamma_n^2\,a\,(t-\tau)]\,\varphi(\tau)\,d\tau\right\}.$$

The functional transformations given above also extend to the corresponding boundary conditions for Eq. (73); the expansion theorems for the transformed equations are of importance for the indicated transformations (references in [16]). Taking into account the promising nature of the direction in further investigations presented in this item, we give below the total data on the types of boundary motion and the form of boundary conditions for which an exact analytical solution of the thermal problem is possible.

Summing up, we emphasize that we are eventually dealing with the necessity of solving spectral problems for the equation

$$\frac{1}{\xi^m}\frac{d}{d\xi}\left(\xi^m\frac{d\Psi(\xi)}{d\xi}\right) + \alpha\,[\gamma^2 - q(\xi)]\,\Psi(\xi) = 0\,, \tag{94}$$

where $q(\xi)$ is a given function on a certain interval of variation of the variable ξ ($q(\xi) = \xi$; ξ^2, etc.); $m = 0$, 1, and 2. These issues are considered in detail in the known monographs of Titchmarsh [231]. As results on the solution of spectral problems of this kind are accumulated, new laws of motion of a boundary, i.e., cases that allow the exact analytical solutions of the corresponding problems of nonstationary heat transfer, can, apparently, appear.

7. Methods of Solution of Problems of Heat Conduction with a Time-Variable Coefficient of Heat Exchange

The indicated class of problems is characteristic of regions with moving boundaries, taking into account that a number of the approaches presented above lead to the necessity of solving the transformed problem with boundary conditions of the form $(\partial W/\partial n)_\Gamma = h(t)[W|_\Gamma - T_{\text{med}})$, where $h(t)$ is a continuously differentiable function. At the same time, this class of problems is of practical interest in regions with fixed boundaries, too, and these cases have begun to receive increased attention in recent decades. The dependence $h(t)$ is observed in the formation of a thermal boundary layer under the conditions of nonstationary flow of cooling water about solid surfaces; in the heating of bodies by a pulsating flow; in the motion of a ballistic body in a medium with variable density and temperature; in the heat exchange of a rolled metal with rolls and the ambient medium; in studying the phenomena of turbulence in contact measuring of the gas temperature; in nonstationary cooling of thermoelectric devices; in diffusion processes at a variable temperature, etc. [154; 232]. In addition to the technological reasons, there are a number of other reasons why the heat-transfer coefficient changes with time: a change in the physical characteristics of a heat-exchange agent (the velocity of motion, the emissivity factor, the density, etc.) or a change in the state of the surface of a heated body with time (oxidation, clogging with dust, cracking, etc.). For an arbitrary law of change in the coefficient $h(t)$, the sought temperature function is not expressed in quadratures and the exact solution of the problem has the form of an infinite series. In practice, one employs different approaches that yield exact (in the form of an infinite series) or approximate solutions of this class of problems for a plate, a cylinder, a sphere, and a semibounded bar for an arbitrary law $h(t)$ and its particular dependences: exponential, power, root, periodic, etc. These are: the method of thermal potentials where the heat-conduction equation is reduced to the Volterra integral equation of the second kind and the Picard process of expansion in a parameter is subsequently used; the integral von Kármán–Pohlhausen method from the theory of a hydrodynamic boundary layer, the method of expansion in a small parameter (perturbation method); the operational approach using the method of successive approximations; the method of a bifrequency transfer function; the method of averaging of functional corrections; the method of reduction of a heat-conduction equation to a system of ordinary differential equations using the Green's function, a variational method; the method of splitting of a generalized integral Fourier transform that yields the integral form of the first approximation for an arbitrary dependence $h(t)$; asymptotic methods and others [119, 233, 234]. Dispite the variety of the approaches, each of them eventually reduces the solution of the problem to an infinite series of successive approximations, and the prime objective of each approach is finding the most appropriate first approximation. Let us briefly consider some of these approaches in finding an analytical solution of the following problem:

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} , \quad x > 0 , \quad t > 0 ; \tag{95}$$

$$T(x, t)\big|_{t=0} = T_0 , \quad x \geq 0 ; \quad |T(x, t)| < +\infty , \quad x \geq 0 , \quad t \geq 0 ; \tag{96}$$

$$(\partial T/\partial x)_{x=0} = h(t)\, T(x, t)\big|_{x=0} , \quad t > 0 . \tag{97}$$

For simplicity of representation, we set $a = 1$ and $T_{\text{med}} = 0$, which does not limit the generality of the consideration. The solution of Eq. (95) is written in the form

$$T(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{+\infty} F(\xi) \exp\left[-\frac{(x-\xi)^2}{4t}\right] d\xi \tag{98}$$

and on the negative semiaxis x as on the initial one we select a function $F(x)$ such that (98) satisfies the boundary condition (97). The latter leads to a functional equation of the form

$$(-1/2)\int_0^{\infty} f(2\sqrt{xt})\exp(-x)\,dx = T_0\,\gamma(t) + \gamma(t)\,(\sqrt{\pi}/2)\int_0^{\infty} f(2\sqrt{xt})x^{-1/2}\,x\exp(-x)\,dx , \tag{99}$$

where $f(x) = F(-x) - T_0$; $\gamma(t) = h(t)\sqrt{\pi t}$. If it is assumed that the function $h(t)$ is expanded into a series in powers $t^{1/2}$, i.e., $\gamma(t) = \sum_{n=0}^{\infty} \gamma_n t^{n/2}$, and the function $f(x)$ is sought in the form of the series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then Eq. (99) yields for the coefficients a_n the relation

$$a_n = \frac{T_0\,\gamma_n + \sum_{m=0}^{n-1} 2^{m-1}\,\Gamma\left(\frac{m+1}{2}\right) a_m\,\gamma_{n-m}/\sqrt{\pi}}{2^{n-1}\,\Gamma\,(1+n/2)},$$

and along with this also the solution $T(x, t)$ in the form

$$T\,(x, t) = T_0 + \frac{1}{2\,\sqrt{\pi t}} \sum_{n=1}^{\infty} a_n \int_0^{\infty} \xi^n \exp\left[-\frac{(x+\xi)^2}{4t}\right] d\xi\,. \tag{100}$$

The method of successive approximations for the equation $\partial T/\partial t = a\partial^2 T/\partial x^2$ with the initial condition (96) and the boundary condition $(\partial T/\partial x)_{x=0} = h(t)\,[T(0, t) - \varphi(t)]$ yields the solution of the problem in another form:

$$T\,(x, t) = \frac{x}{2\,\sqrt{a\pi}} \int_0^t \frac{A\,(\tau)}{(t-\tau)^{3/2}} \exp\left[-\frac{x^2}{4a\,(t-\tau)}\right] d\tau\,, \tag{101}$$

$$A\,(t) = \sum_{n=0}^{\infty} (-1)^n \left(\sqrt{\frac{a}{\pi}}\right)^{n+1} \int_0^t \frac{h\,(\tau)\,d\tau}{\sqrt{t-\tau}} \int_0^{\tau} \frac{h\,(\tau_1)}{\sqrt{\tau-\tau_1}} \dots \int_0^{\tau_{n-1}} \frac{h\,(\tau_n)\,\varphi\,(\tau_n)}{\sqrt{\tau_{n-1}-\tau_n}}\,d\tau_n\,.$$

For the function $h(t)$ bounded on the segment $[0, t]$, the series (101) converges absolutely and uniformly for all $x > 0$ and $t > 0$ in any finite interval of their variation and allows a number of particular cases of interest. Thus, for $h(t) = h_0 t^m$ and $\varphi(t) = T_{\text{med}} t^r$, where m and r are real numbers, expression (101) takes the form

$$T\,(x, t) = \frac{T_{\text{med}} x}{2\,\sqrt{a\pi}} \sum_{n=1}^{\infty} (-1)^{n+1} \left(h_0 \sqrt{\frac{a}{\pi}}\right)^n \prod_{k=1}^{n} B\,(r + 1/2 + k\,(m+1/2)\,;\,1/2)\,\times$$

$$\times \int_0^t \frac{\tau^{r+n(m+1/2)}}{(t-\tau)^{3/2}} \exp\left[-\frac{x^2}{4\,(t-\tau)}\right] d\tau\,, \tag{102}$$

where $B(c, d)$ is the beta function, and it is assumed that $[r + 1/2 + k(m + 1/2)] > 0$. For $m = -(1/2)$, i.e., $h(t) = h_0 t^{-1/2}$ and $r = 0$, expression (102) yields the compact solution

$$T\,(x, t) = T_{\text{med}}\,\Phi^*\left(\frac{x}{2\,\sqrt{at}}\right) \sum_{n=1}^{\infty} (-1)^{n+1}\,(h_0\,\sqrt{a\pi}\,)^n = \frac{T_{\text{med}} h_0\,\sqrt{a\pi}}{1 + h_0\,\sqrt{a\pi}}\,\Phi^*\left(\frac{x}{2\,\sqrt{at}}\right)$$

on condition that $h_0\sqrt{a\pi} < 1$. Similarly, we can also consider the remaining cases. In particular, problem (95)–(97) allows the exact solution for $h(t) = h_0(At^2 + Bt + C)^{-1/2}$ if, to find the solution, we use the functional transformations presented in item 6 ($z = x/y(t)$, etc.). As another example having numerous applications, in (95)–(97) we consider the case where the condition of heat exchange (97) is assigned in the form of a convolution of two functions:

$$(\partial T/\partial x)_{x=0} = \int_0^t h(t-\tau)\, T(0, \tau)\, d\tau\,, \quad t > 0\,. \tag{103}$$

In the space of (Laplace) transforms, the solution has the form

$$\frac{\overline{T}(x, p)}{T_0} = \frac{1}{p} + \frac{1}{p} \exp(-x\sqrt{p}\,) \sum_{n=0}^{\infty} (-1)^{n+1} \left[\frac{\overline{h}(p)}{\sqrt{p}}\right]^{n+1},$$

whence it is easy to write two approximations for the inverse transform:

$$T(x, t)/T_0 = 1 - \frac{1}{\sqrt{\pi}} \int_0^t \Phi\left(\frac{x}{2\sqrt{t-\tau}}\right) d\tau \int_0^\tau \frac{h(\tau_1)}{\sqrt{\tau - \tau_1}}\, d\tau_1 +$$

$$+ \left[\frac{1}{\sqrt{\pi}} \int_0^t \Phi\left(\frac{x}{2\sqrt{t-\tau}}\right) d\tau \int_0^\tau \frac{h(\tau_1)}{\sqrt{\tau-\tau_1}}\, d\tau_1\right]_t * \left[\frac{1}{\sqrt{\pi}} \int_0^t \frac{h(\tau)\, d\tau}{\sqrt{t-\tau}}\right] + \dots\,.$$

Thus, the above approaches yield different functional expressions for the first terms of an infinite series of successive approximations, and the analytical solution of the problem in closed form can be obtained just for a small number of the particular dependences $h(t)$ in (97). The extension of a class of dependences $h(t)$ of this kind is one of the open problems of the analytical theory of heat conduction for boundary-value problems of nonstationary heat and mass transfer with a time-variable relative coefficient of heat exchange (mass exchange).

Dispite the apparent simplicity of mathematical models of nonstationary transfer in regions with moving boundaries, the given problems are far from being trivial for obtaining their exact analytical solution. A practical study is made of the simplest laws of motion of a boundary (linear, parabolic, and quadratic laws), but for these cases, too, the analytical theory of heat conduction is only in the beginning stages of its development. With regard to the more complex laws of motion of a boundary that are indicated above in Table 1 and allow the exact analytical solutions of the corresponding boundary-value problems, there is still much work to be done to find these solutions, study their properties, and construct temperature nomograms. As the analysis of literature sources shows, solution of problems of this kind brings up a large class of problems of computational mathematics, the theory of special functions, and the methods of mathematical physics. Also, there is much to be done with regard to the simplest problems of transfer in canonical regions, dispite the well-developed theory for these cases. The task of a researcher is to see these problems and to understand the necessity of studying them. The expediency of finding exact analytical solutions for problems of heat conduction in all the cases where it is possible is also obvious. The exact analytical solutions yield graphical dependences between the parameters of the processes in question that are more general, clear, and appropriate to the phenomenon within the framework of the adopted model than the dependences established in numerical solution of the initial problem or on the basis of its approximate solution. In the

TABLE 1. Types of Motion of a Boundary and the Form of Boundary Conditions for the Analytical Solution of a Thermal Problem

	Region	Moving boundary	Boundary conditions are realized
Cartesian coordinates	$(At+B)^2 + C \le x < \infty$	$\xi = x - y(t)$	*I*; *II*
	$\pm\sqrt{(At+B)^2 + C} \le x < \infty$	$\xi = x/y(t)$	*I*; *III*: $h(t) = y'(t)/(2a)$
	$\beta \le x \le (At+B)^2 + C$	$\xi = \dfrac{x - y_1(t)}{y_2(t) - y_1(t)}$	*I*; *II*$(\beta = 0)$
	$\beta\sqrt{(At+B)^2 + C} \le x \le \sqrt{(At+B)^2 + C}$	$\xi = x/y(t)$	*I*; *II*$(\beta = 0)$; *III*: $\beta = 0 \cap h(t) \sim y^{-1}(t)$; $\beta < 1 \cap h(t) = \beta y'(t)/(2a)$
	$\beta(C \pm \sqrt{At+B}) \le x \le C \pm \sqrt{At+B}$	$\xi = \dfrac{x - y_1(t)}{y_2(t) - y_1(t)}$	*I*; *II*$(\beta = 0)$
	$(At+B)^2 + C \le x \le (At+B)^2 + C + D$	$\xi = \dfrac{x - y_1(t)}{y_2(t) - y_1(t)}$	*I*
	$(At+B)^{-1}C_1 + C_2 t + C_3 \le x \le (At+B)^{-1}$ $C_1 + (C_2 + A)t + C_3 + B$	$\xi = \dfrac{x - y_1(t)}{y_2(t) - y_1(t)}$	*I*
	$C_1\sqrt{(At+B)^2 + C} \le x \le C_4$ $\sqrt{(At+B)^2 + C} + C_2 t + C_3$	$\xi = \dfrac{x - y_1(t)}{y_2(t) - y_1(t)}$	*I*
Cylindrical coordinates	$\sqrt{(At+B)^2} + C \le r < \infty$	$\xi = r/y(t)$	*I*; *III*: $h(t) = y'(t)/(2a)$
	$At + B \le r < \infty$	$\xi = r/y(t)$	*I*; *III*: $h(t) = y'(t)/(2a)$
	$\beta\sqrt{(At+B)^2 + C} \le r \le$ $\le \sqrt{(At+B)^2 + C}$ $(0 \le \beta < 1)$	$\xi = r/y(t)$	*I*; *II*$(\beta = 0)$; *III*: $\beta = 0 \cap h(t) \sim y^{-1}(t)$; $0 < \beta < 1 \cap h(t) = y'(t)/(2a)$
	$\beta(At+B) \le r \le At + B$ $(0 \le \beta < 1)$	$\xi = r/y(t)$	*I*; *II*$(\beta = 0)$; *III*: $\beta = 0 \cap h(t) \sim y^{-1}(t)$; $0 < \beta < 1 \cap h(t) = \beta y'(t)/(2a)$

Note: $h(t)$ is the relative coefficient of heat transfer in the boundary condition of the 3rd kind.

latter case, one traditionally comes across the serious problem of evaluating the error of the obtained result that most frequently remains unknown because of the difficulties of a computational character. It is also significant that solutions known in the literature refer to thermal problems based on the classical Fourier phenomenology of the propagation of heat in solids, i.e., for parabolic equations. For more complicated transport equations based on the hypothesis of Maxwell–Cattaneo–Luikov (a hyperbolic equation [11]), in media with thermal memory (based on a linearized theory [151]) in deformable media with allowance for the effect of connectedness of the field of temperature and deformation in a heat-conduction equation [16], etc., the indicated cases represent a practically undeveloped area of the analytical theory of nonstationary transport. In this area, there can, probably, be physical results of interest concerning the influence of the motion of a boundary on transfer processes. These investigations will be in the XXIst century, and the author wishes the readers success along this road!

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